ECE 308 - 03
Introduction to Discrete Time Signals & Systems
HW #4

February 4, 2002

3.2. Determine the z-transform of the following signals and sketch the corresponding pore-zero pattern.
(40 points)

a.

$$x(n) = (1+n)u(n)$$

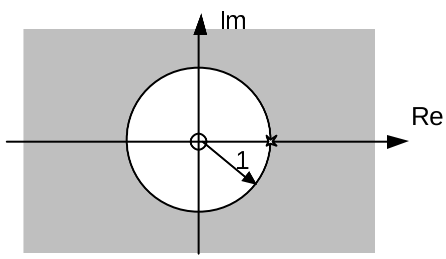
$$x(n) = u(n) + nu(n)$$

From the Table 3.3 Z-transform of x(n) function

$u(n) \leftrightarrow \dfrac{1}{1-z^{-1}}$ ROC: $|z| > 1$ $\qquad nu(n) \leftrightarrow \dfrac{z^{-1}}{(1-z^{-1})^2}$ ROC: $|z| > 1$

$$X(z) = \frac{1}{1-z^{-1}} + \frac{z^{-1}}{(1-z^{-1})^2}, \quad \text{ROC: } |z| > 1$$

$$X(z) = \frac{1-z^{-1}+z^{-1}}{(1-z^{-1})^2} = \frac{1}{(1-z^{-1})^2}, \quad \text{ROC: } |z| > 1$$



b.

$$x(n) = \left(a^n + a^{-n}\right)u(n), \text{ a is real}$$

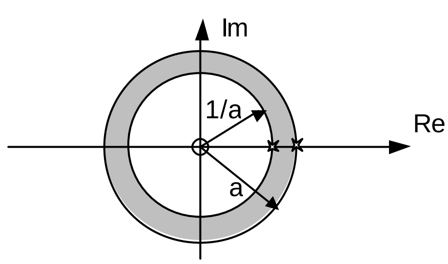
$$x(n) = a^n u(n) + a^{-n} u(n)$$

$$\sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} \quad \text{ROC: } |z| > |a|$$

$$\sum_{n=0}^{\infty} a^{-n} z^{-n} = \sum_{n=0}^{\infty} (a^{-1}z^{-1})^n = \frac{1}{1-a^{-1}z^{-1}} \quad \text{ROC: } |z| > \left|\frac{1}{a}\right|$$

$$X(z) = \frac{1}{1-az^{-1}} + \frac{1}{1-a^{-1}z^{-1}} \quad \text{ROC: } |z| > \left|\frac{1}{a}\right| \text{ if } |a| < 1 \text{ or } |z| > |a| \text{ if } |a| > 1$$

$$X(z) = \frac{1-a^{-1}z^{-1} + (1-az^{-1})}{(1-az^{-1})(1-a^{-1}z^{-1})} = \frac{2-(a^{-1}+a)z^{-1}}{(1-a^{-1}z^{-1})(1-a^{-1}z^{-1})}$$



e **c**

$$x_1(n) = (a^n \cos w_0 n)u(n)\,,\quad X_1(z) = \frac{1 - az^{-1}\cos w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-2}}\,,\quad \text{ROC: } |z| > |a|$$

$$x(n) = nx_1(n)\quad X(z) = -z\frac{d}{dz}X_1(z) = -z\frac{d}{dz}\left(\frac{1 - az^{-1}\cos w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-2}}\right)$$

$$X(z) = -z\left(\frac{az^{-2}\cos w_0(1 - 2az^{-1}\cos w_0 + a^2 z^{-2}) - (2az^{-2}\cos w_0 - 2a^2 z^{-3})(1 - az^{-1}\cos w_0)}{(1 - 2az^{-1}\cos w_0 + a^2 z^{-2})^2}\right)$$

$$X(z) = -z\left(\frac{az^{-2}\cos w_0 - 2a^2 z^{-3}\cos^2 w_0 + a^3 z^{-4}\cos w_0 - 2az^{-2}\cos w_0 + 2a^2 z^{-3}\cos^2 w_0 + 2a^2 z^{-3} - 2a^3 z^{-4}\cos w_0}{(1 - 2az^{-1}\cos w_0 + a^2 z^{-2})^2}\right)$$

$$X(z) = -z\left(\frac{-(az^{-2} + a^3 z^{-4})\cos w_0 + 2a^2 z^{-3}}{(1 - 2az^{-1}\cos w_0 + a^2 z^{-2})^2}\right)$$

$$X(z) = \frac{(az^{-1} + a^3 z^{-3})\cos w_0 - 2a^2 z^{-2}}{(1 - 2az^{-1}\cos w_0 + a^2 z^{-2})^2}\,,\quad \text{ROC: } |z| > |a|$$

h.

$$x(n) = \left(\frac{1}{2}\right)^n [u(n) - u(n-10)]$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-10)$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^{(n-10+10)} u(n-10)$$

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{(n-10)} u(n-10)$$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} - \left(\frac{1}{2}\right)^{10}\frac{z^{-10}}{1 - \frac{1}{2}z^{-1}}\qquad \text{ROC: } |z| > \frac{1}{2}$$

$$X(z) = \frac{1 - \left(\frac{1}{2}\right)^{10} z^{-10}}{1 - \frac{1}{2}z^{-1}}\qquad \text{ROC: } |z| > \frac{1}{2}$$

3.3a Determine the z-transform and sketch the ROC of the following signal

$$x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & n \ge 0 \\ \left(\frac{1}{2}\right)^{-n} & n < 0 \end{cases},$$

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n z^{-1} = \frac{1}{1-\frac{1}{3}z^{-1}}, \text{ ROC: } |z| > \frac{1}{3}$$

$$\sum_{n=-\infty}^{1}\left(\frac{1}{2}\right)^{-n} z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n} z^{n} - 1 = \frac{1}{1-\frac{1}{2}z} - 1 = \frac{1-(1-\frac{1}{2}z)}{1-\frac{1}{2}z} = \frac{\frac{1}{2}z}{1-\frac{1}{2}z} = -\frac{1}{1-2z^{-1}}, \text{ ROC: } |z| < 2$$

$$X(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{1}{1-2z^{-1}} = \frac{(1-2z^{-1})-(1-\frac{1}{3}z^{-1})}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} = \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}, \text{ ROC: } \frac{1}{3} < |z| < 2$$

3.7 Compute the convolution of the following signals by means of the z-transform

$$x_1(n) = \begin{cases} \left(\frac{1}{3}\right)^n & n \ge 0 \\ \left(\frac{1}{2}\right)^{-n} & n < 0 \end{cases} \qquad x_2(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$X(z) = \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} \text{ ROC: } \frac{1}{3} < |z| < 2, \quad X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \text{ ROC: } |z| > \frac{1}{2}$$

ROC: $X(z) = X_1(z)X_2(z) = \dfrac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}\dfrac{1}{(1-\frac{1}{2}z^{-1})}$

$$X(z) = \frac{-\frac{5}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})(1-\frac{1}{2}z^{-1})} = \frac{-\frac{5}{3}z^{2}}{(z-\frac{1}{3})(z-2)(z-\frac{1}{2})}$$

$$\frac{X(z)}{z} == \frac{-\frac{5}{3}z}{(z-\frac{1}{3})(z-2)(z-\frac{1}{2})} = \frac{A_1}{(z-\frac{1}{3})} + \frac{A_2}{(z-2)} + \frac{A_3}{(z-\frac{1}{2})}$$

$$A_1 = \left.\frac{(z-\frac{1}{3})X(z)}{z}\right|_{z=\frac{1}{3}} = \left.\frac{-\frac{5}{3}z}{(z-2)(z-\frac{1}{2})}\right|_{z=\frac{1}{3}} = \frac{-\frac{5}{3}\frac{1}{3}}{(\frac{1}{3}-2)(\frac{1}{3}-\frac{1}{2})} = 2$$

$$A_2 = \left.\frac{(z-2)X(z)}{z}\right|_{z=2} = \left.\frac{-\frac{5}{3}z}{(z-\frac{1}{3})(z-\frac{1}{2})}\right|_{z=2} = \frac{-\frac{5}{3}2}{(2-\frac{1}{3})(2-\frac{1}{2})} = -\frac{4}{3}$$

$$A_3 = \left.\frac{(z-\frac{1}{2})X(z)}{z}\right|_{z=\frac{1}{2}} = \left.\frac{-\frac{5}{3}z}{(z-\frac{1}{3})(z-2)}\right|_{z=\frac{1}{2}} = \frac{-\frac{5}{3}\frac{1}{2}}{(\frac{1}{2}-\frac{1}{3})(\frac{1}{2}-2)} = \frac{10}{3}$$

$$\frac{X(z)}{z} = \frac{2}{(z-\frac{1}{3})} - \frac{\frac{4}{3}}{(z-2)} + \frac{\frac{10}{3}}{(z-\frac{1}{2})}$$

$$X(z) = \frac{2}{(1-\frac{1}{3}z^{-1})} - \frac{\frac{4}{3}}{(1-2z^{-1})} + \frac{\frac{10}{3}}{(1-\frac{1}{2}z^{-1})}$$

$$x(n) = 2\left(\frac{1}{3}\right)^n u(n) + \frac{10}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3}\left(\frac{1}{2}\right)^{-n} u(-n-1)$$

3.11 Using long division, determine the inverse z-transform

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

a. x(n) is causal

$$\begin{array}{r|l} & 1+4z^{-1}+7z^{-2}+10z^{-3}+13z^{-4}+\ldots \\ 1-2z^{-1}+z^{-2} & 1+2z^{-1} \\ & \underline{1-2z^{-1}+z^{-2}} \\ & 4z^{-1}-z^{-2} \\ & \underline{4z^{-1}-8z^{-2}+4z^{-3}} \\ & 7z^{-2}-4z^{-3} \\ & \underline{7z^{-2}-14z^{-3}+7z^{-4}} \\ & 10z^{-3}-7z^{-4} \\ & \underline{10z^{-3}-20z^{-4}+10z^{-5}} \\ & 13z^{-4}-10z^{-5} \end{array}$$

$$x(n) = \{\underset{\uparrow}{1},4,7,10,13,\ldots.3n+1\ldots\}$$

b. X(n) is anticausal

$$\begin{array}{r|l} & 2z+5z^{2}+8z^{3}+11z^{4}+14z^{5}\ldots \\ z^{-2}-2z^{-1}+1 & 2z^{-1}+1 \\ & \underline{2z^{-1}-4+2z} \\ & 5-2z \\ & \underline{5-10z+5z^{2}} \\ & 8z-5z^{2} \\ & \underline{8z-16z^{2}+8z^{3}} \\ & 11z^{2}-8z^{3} \\ & \underline{11z^{2}-22z^{3}+11z^{4}} \\ & 14z^{3}-11z^{4} \end{array}$$

$$x(n) = \{\ldots,-(3n+1),\ldots,14,11,8,5,1,0\}$$

$\uparrow$