Example:

$$x_a(t) = 3\cos 100\pi t$$

a. Find the minimum sampling rate required to avoid aliasing.
b. If $F_s = 200\ Hz$, What is the discrete-time signal after sampling?
c. If $F_s = 75 Hz$, What is the discrete-time signal after sampling?
d. What is the frequency F of a sinusoidal that yields sampling identical to obtained in part c?

Solution:

a. $\Omega = 100\pi \rightarrow F = 50\ \mathrm{Hz}$

The minimum sampling rate is

$$F_s = 2F = 100\,\mathrm{Hz}$$

and the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{100}n = 3\cos\pi n$$

b. If $F_s = 200\ Hz$, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$$

c. If $F_s = 75\ \mathrm{Hz}$, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n$$

$$= 3\cos\left(2\pi - \frac{2\pi}{3}\right)n$$

$$= 3\cos\frac{2\pi}{3}n$$

d. For the sampling rate $F_s = 75\text{ Hz}$,

$F = fF_s = f75$, and $f = \frac{1}{3}$ in part in (c). Hence

$F = \frac{75}{3} = 25\text{Hz}$

So, the analog sinusoidal signal is

$$y_a(t) = 3\cos 2\pi Ft$$

$$= 3\cos 50\pi t$$

The Sampling Theorem

We must have some information about the analog signal especially the frequency content of the signal, to select the sampling period T or sampling rate F_s .

For example: A speech signal goes below around 3Khz.
A TV signal is up to 5Mhz.

Any analog signal can be represented as sum of sinusoids of different amplitudes, frequencies, and phases.

$$x_a(t) = \sum_{i=1}^{N} A_i \cos(2\pi F_i t + \theta_i)$$

where N the number of frequency components. Suppose that Nth frequency do not exceed the largest frequency $F_{\max}$

$$|F_i| < F_{\max}$$

To avoid the aliasing problem, F_s is selected so that

$$F_s > 2F_{\max}$$

The analog signal should be in the range of

$$-\frac{1}{2} \le f_i = \frac{F_i}{F_s} \le \frac{1}{2}$$

or in radians

$$-\pi \le \omega_i = 2\pi f_i \le \pi$$

The sampling rate $F_N = 2F_{\max}$ is called the *Nyquist rate*.

Example:

Consider an analog signal

$$x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t + 3\cos 100\pi t$$

The frequencies in the analog signal

$F_1 = 25\,\text{Hz}$, $F_2 = 150\,\text{Hz}$, $F_3 = 50\,\text{Hz}$

The largest frequency is

$F_{\max} = F_2 = 150\,\text{Hz}$

The Nyquist rate is

$F_N = 2F_{\max} = 300\ \text{Hz}$

Example:

The analog signal

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t - 10\cos 12000\pi t$$

a. What is the Nyquist rate for this signal?
b. Using a sampling rate $F_s = 5000\ \text{samples/s}$. What is the discrete-time signal obtained after sampling?
c. What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation?

Solution:

a. The frequencies of the analog signal are

$F_1 = 1\ \text{KHz}$, $F_2 = 3\ \text{KHz}$, $F_3 = 6\,\text{KHz}$

The Nyquist rate is $F_N = 2F_{\max} = 12\ \text{KHz}$

b. For $F_s = 5\text{KHz}$

$$x(n) = x_a(nT) = x_a\left(\frac{n}{F_a}\right)$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(\frac{3}{5}\right)n + 10\cos 2\pi\left(\frac{6}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(1-\frac{2}{5}\right)n + 10\cos 2\pi\left(1+\frac{1}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(-\frac{2}{5}\right)n + 10\cos 2\pi\left(\frac{1}{5}\right)n$$

$$= 13\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n$$

For $F_s = 5\,\mathrm{KHz}$, the folding frequency is $F_{\max} = \frac{F_s}{2} = 2.5\,\mathrm{KHz}$

Hence, $F_1 = 1\,\mathrm{KHz}$ is not effected by aliasing.

$F_2 = 3\,\mathrm{KHz}$ is changed by the aliasing effect

$$F_2^{'} = F_2 - F_s = -2\,\mathrm{KHz}$$

$F_3 = 6\,\mathrm{KHz}$ is changed by the aliasing effect

$$F_3^{'} = F_3 - F_s = 1\,\mathrm{KHz}$$

So that normalize frequencies are $f_1 = \frac{1}{5},\ f_2 = -\frac{2}{5},\ \ f_3 = \frac{1}{5}$

c. The analog signal we can recover is

$$y_a(t) = 13\cos 2000\pi t - 5\cos 4000\pi t$$

which is different than the original signal $x_a(t)$.