Solution of Difference Equation:

The one-sided z-transform is a very efficient tool for the solution of difference equation.

Example:
Let y(n) represents the Fibonnacci series
1,1,2,3,5,8,13,21,…..

$$y(n) = y(n-1) + y(n-2)$$

The initial condition

$$y(0) = y(-1) + y(-2) = 1$$
$$y(1) = y(0) + y(-1) = 1 \qquad \rightarrow y(-1) = 0,\ y(-2) = 1$$

$$Y^+(z) = z^{-1}\left[Y^+(Z) + y(-1)z\right] + z^{-2}\left[Y^+(z) + y(-1)z + y(-2)z^2\right]$$
$$Y(z) = z^{-1}Y^+(Z) + y(-1) + z^{-2}Y^+(z) + z^{-1}y(-1) + y(-2)$$
$$Y(z)(1 - z^{-1} - z^{-2}) = 1$$
$$Y(z) = \frac{1}{1 - z^{-1} - z^{-2}} = \frac{z^2}{z^2 - z - 1}$$
$$\frac{Y(z)}{z} = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)} = \frac{A_1}{\left(z - \frac{1+\sqrt{5}}{2}\right)} + \frac{A_2}{\left(z - \frac{1-\sqrt{5}}{2}\right)}$$
$$A_1 = \left.\frac{z}{\left(z - \frac{1-\sqrt{5}}{2}\right)}\right|_{z=\frac{1+\sqrt{5}}{2}} = \frac{\frac{1+\sqrt{5}}{2}}{\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)} = \frac{\frac{1+\sqrt{5}}{2}}{\sqrt{5}}$$

$$A_2 = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)}\Bigg|_{z=\frac{1-\sqrt{5}}{2}} = \frac{\frac{1-\sqrt{5}}{2}}{\left(\frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}\right)} = \frac{-\left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}}$$

$$y(n) = \left[\frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]u(n)$$

Example:

Find the step response of the system

$y(n) = a\, y(n-1) + x(n) \qquad -1 < a < 1$

The initial condition $y(-1) = 1$

Solution:

Taking one-sided z-transform

$$Y^+(z) = a\left[z^{-1}Y^+(Z) + y(-1)\right] + X^+(z)$$

$$Y^+(z) = a\, z^{-1}Y^+(Z) + a\, y(-1) + \frac{1}{1-z^{-1}}$$

$$Y^+(z) = \frac{a}{1-a\, z^{-1}} + \frac{1}{(1-a\, z^{-1})(1-z^{-1})}$$

The inverse transform

$$y(n) = a^{n+1}u(n) + \frac{1-a^{n+1}}{1-a}u(n)$$

$$= \frac{1}{1-a}(1-a^{n+2})u(n)$$

3.6 Analysis of LTI Systems in the z-Domain

a. <u>Response of System with Rational System Function</u>

Let us assume that the input signal x(n) and the corresponding system function h(n) have rational z-transform X(z) and H(z) of the form

$$X(z) = \frac{N(z)}{D(z)} \quad \text{and} \quad H(z) = \frac{B(z)}{A(z)}$$

If the system initially relaxed (initial conditions for difference equation are zero), the z-transform of system output has

$$Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)D(z)}$$

Partial fraction expansion of Y(z) will be in the following form if no pole-zero cancellation

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_x z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_x z^{-1}}$$

where $p_{1,} p_2, \ldots p_N$ system poles, $q_{1,} q_2, \ldots q_L$ the input signal poles, and $p_k \neq q_m$

The inverse transform of Y(z) yields

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

where A_k and Q_k are functions of both sets of poles p_k and q_k .

We can separate the y(n) into two parts

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$$ → <u>natural response</u>

Natural response is different than zero-input response. If X(z) is zero, then Y(z) is zero.

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$ → <u>forced response</u> of the system.

b. <u>Response of Pole-zero System with Nonzero Initial Condition.</u>

The input signal x(n) is assumed to be causal. The effect of all previous input signals to the system are reflected in the initial conditions y(-1), y(-2), …..y(-N). We will look at the one-sided z-transform

$$Y^+(z) = -\sum_{k=1}^{N} a_k z^{-k}\left[Y^+(z) + \sum_{k=1}^{k} y(-n) z^n\right] + \sum_{k=1}^{M} b_k z^{-k} X^+(z)$$

Since x(n) is causal, we can set $X^+(z) = X(z)$

$$Y^+(z) = \frac{\sum_{k=1}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z) - \frac{\sum_{k=1}^{N} a_k z^{-k} \sum_{k=1}^{k} y(-n) z^n}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$Y^+(z) = H(z) X(z) - \frac{N_0(z)}{A(z)}$$

$$N_0(z) = \sum_{k=1}^{N} a_k z^{-k} \sum_{k=1}^{k} y(-n) z^n$$

The output of the system can subdivide into two parts.

$Y_{zs}(z) = H(z) X(z)$ ->The zero-state response

$$Y_{zi}^{+}(z) = \frac{N_0(z)}{A(z)}$$ → the zero-input response

Therefore, the total of the inverse z-transform of responses gives us y(n)

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

Example:
Determine the unit step response of the system of the following equation.

$$y(n) = 0.9y(n-1) - 0.81y(n-2) + x(n)$$

Condition:
a. $y(-1) = y(-2) = 0$
b. $y(-1) = y(-2) = 1$

Solution:

$$Y(z) = 0.9\left[z^{-1}Y(z) + y(-1)\right] - 0.81\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] + X(z)$$

$$Y(z)(1 - 0.9z^{-1} + 0.81z^{-2}) = 0.9y(-1) - 0.81z^{-1}y(-1) + 0.81y(-2) + X(z)$$

$$Y(z) = \frac{1}{(1 - 0.9z^{-1} + 0.81z^{-2})}X(z)$$

The system function is

$$H(z) = \frac{1}{(1 - 0.9z^{-1} + 0.81z^{-2})}$$

The system poles are

$p_1 = 0.9e^{j\frac{p}{3}}$ and $p_2 = 0.9e^{-j\frac{p}{3}}$

The z-transform of input sequence is

$$X(z) = \frac{1}{(1 - z^{-1})}$$

Therefore,

$$Y(z) = \frac{1}{(1-0.9e^{j\frac{p}{3}}z^{-1})(1-0.9e^{j\frac{p}{3}}z^{-1})(1-z^{-1})}$$

$$Y(z) = \frac{0.544e^{-j95.2}}{(1-0.9e^{j\frac{p}{3}}z^{-1})} + \frac{0.544e^{-j95.2}}{(1-0.9e^{j\frac{p}{3}}z^{-1})} + \frac{1.099}{(1-z^{-1})}$$

The zero-state response is

$$y_{zs}(n) = \left[1.099 + 1.088(0.9)^n \cos(\frac{p}{3}n - 95.2^\circ)\right]u(n)$$

Since the initial condition are zero , $y(n) = y_{zs}(n)$

c. Using initial condition

$$Y(z)(1-0.9z^{-1}+0.81z^{-2}) = 0.9-0.81z^{-1}+0.81+X(z)$$

$$Y(z) = \frac{1}{(1-0.9z^{-1}+0.81z^{-2})}X(z) - \frac{0.81-0.81z^{-1}}{(1-0.9z^{-1}+0.81z^{-2})}$$

$$Y_{zi}(z) = \frac{N_0(z)}{A(z)} = \frac{0.81-0.81z^{-1}}{(1-0.9z^{-1}+0.81z^{-2})}$$

$$Y_{zi}(z) = \frac{0.4956e^{j84.8}}{1-0.9e^{j\frac{p}{3}}z^{-1}} + \frac{0.4956e^{-j84.8}}{1-0.9e^{j\frac{p}{3}}z^{-1}}$$

The zero input response

$$y_{zi}(n) = \left[0.99(0.9)^n \cos(\frac{p}{3}n + 84.8^\circ)\right]u(n)$$

The total response

$$Y(z) = Y_{zs}(z) + Y_{zi}(z)$$

$$y_{zs}(n) = \left[1.099 + 1.44(0.9)^n \cos(\frac{p}{3}n + 38^\circ)\right]u(n)$$