

The One-Side z-Transform

The one-sided z-transform of a signal $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$x(n) \stackrel{z^+}{\leftrightarrow} X^+(z)$$

The one-sided z-transform has the following characteristics:

1. It does not contain information about the signal $x(n)$ for negative values of time (i.e., for $n < 0$)
2. It is unique only for causal signals, because only these signals are zero for $n < 0$.
3. The one-sided z-transform $X^+(z)$ of $x(n)$ is identical to the two-sided z-transform of the signal $x(n)u(n)$.
4. ROC of $X^+(z)$ is always the exterior of the circle. So it is not necessary to refer to their ROC.

Example:

$$x_1(n) = \{1, 2, 5, 7, 0, 1\} \stackrel{x^+}{\leftrightarrow} X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

↑

$$x_2(n) = \{1, 2, 5, 7, 0, 1\} \stackrel{z^+}{\leftrightarrow} X_2^+(z) = 5 + 7z^{-1} + z^{-3}$$

↑

$$x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\} \stackrel{x^+}{\leftrightarrow} X_3^+(z) = 1z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

↑

$$x_4(n) = \mathbf{d}(n) \stackrel{x^+}{\leftrightarrow} X_4^+(z) = 1, \quad x_5(n) = \mathbf{d}(n-k) \stackrel{x^+}{\leftrightarrow} X_5^+(z) = z^{-k}$$

$$x_6(n) = \mathbf{d}(n+k) \stackrel{x^+}{\leftrightarrow} X_6^+(z) = 0$$

$$x_4(n) = a^n u(n) \stackrel{x^+}{\leftrightarrow} X_4^+(z) = \frac{1}{1-az^{-1}}$$

Shifting Property

Case 1: Time Delay

If $x(n) \stackrel{x^+}{\leftrightarrow} X^+(z)$

Then

$$x(n-k) \stackrel{x^+}{\leftrightarrow} z^{-k} \left[X^+(z) + \sum_{n=1}^k x(-n)z^n \right] \quad k > 0$$

In case $x(n)$ is causal, then

$$x(n-k) \stackrel{x^+}{\leftrightarrow} z^{-k} X^+(z) \quad k > 0$$

Example:

$$x(n) = a^n u(n) \stackrel{x^+}{\leftrightarrow} X^+(z) = \frac{1}{1-az^{-1}}$$

The z-transform of $x_1(n) = x(n-2)$

$$\begin{aligned} x(n-2) = a^{n-2} u(n-2) &\stackrel{x^+}{\leftrightarrow} z^{-2} \left[\frac{1}{1-az^{-1}} + x(-1)z + x(-2)z^2 \right] \\ &= \frac{z^{-2}}{1-az^{-1}} + x(-1)z^{-1} + x(-2) \end{aligned}$$

Case 2: Time advance

If $x(n) \stackrel{x^+}{\leftrightarrow} X^+(z)$

Then

$$x(n+k) \stackrel{x^+}{\leftrightarrow} z^k \left[X^+(z) - \sum_{n=0}^{k-1} x(n)z^{-n} \right] \quad k > 0$$

Example:

$$x(n) = a^n u(n) \xleftrightarrow{x^+} X^+(z) = \frac{1}{1 - az^{-1}}$$

The z-transform of $x_1(n) = x(n+2)$

$$\begin{aligned} x(n-2) = a^{n-2} u(n-2) &\xleftrightarrow{x^+} z^2 \left[\frac{1}{1 - az^{-1}} + x(0) + x(1)z^{-1} \right] \\ &= \frac{z^2}{1 - az^{-1}} + x(0)z^2 + x(1)z \\ &= \frac{z^2}{1 - az^{-1}} + z^2 + az \end{aligned}$$

Final Value Theorem:

$$\text{If } x(n) \xleftrightarrow{x^+} X^+(z)$$

Then

$$\lim_{n \rightarrow \infty} x(n) \xleftrightarrow{x^+} \lim_{z \rightarrow 1} (z-1)X^+(z)$$

This exists if the ROC of $(z-1)X^+(z)$ includes the unit circle.

Example:

$$x(n) = u(n), \quad h(n) = a^n u(n)$$

$$y(n) = h(n) * x(n)$$

$$Y(z) = H(z)X(z) = \frac{1}{1 - az^{-1}} \frac{1}{1 - z^{-1}} = \frac{z^2}{(z-a)(z-1)} \quad \text{ROC: } |z| > |a|$$

$$(z-1)Y(z) = (z-1) \frac{z^2}{(z-a)(z-1)} = \frac{z^2}{z-a} \quad \text{ROC: } |z| > |a|$$

$$\lim_{n \rightarrow \infty} x(n) \stackrel{x^+}{\leftrightarrow} \lim_{z \rightarrow 1} \frac{z^2}{z-a} = \frac{1}{1-a}$$

Solution of Difference Equation:

The one-sided z-transform is a very efficient tool for the solution of difference equation.

Example:

Let $y(n_0)$ represents the Fibonacci series

1, 1, 2, 3, 5, 8, 13, 21,

$$y(n) = y(n-1) + y(n-2)$$

The initial condition

$$y(0) = y(-1) + y(-2) = 1$$

$$y(1) = y(0) + y(-1) = 1 \quad \rightarrow y(-1) = 0, \quad y(-2) = 1$$

$$Y(z) = z^{-1} [Y^+(z) + y(-1)] + z^{-2} [Y^+(z) + y(-1) + y(-1)z^{-1}]$$

$$Y(z) = \frac{1}{1-z^{-1}-z^{-2}} = \frac{z^2}{z^2-z-1}$$

$$\frac{Y(z)}{z} = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)} = \frac{A_1}{\left(z - \frac{1+\sqrt{5}}{2}\right)} + \frac{A_2}{\left(z - \frac{1-\sqrt{5}}{2}\right)}$$

$$A_1 = \frac{z}{\left(z - \frac{1-\sqrt{5}}{2}\right)} \Bigg|_{z=\frac{1+\sqrt{5}}{2}} = \frac{1+\sqrt{5}}{\sqrt{5}}$$

$$A_2 = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)} \Bigg|_{z=\frac{1-\sqrt{5}}{2}} = \frac{-\left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}}$$

$$y(n) = \left[\frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^2 - \frac{1-\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^2 \right] u(n)$$