<u>The inverse z-transform by Partial Fraction Expansion</u>

<u>Multiple – order poles:</u>
If X(z) has repeated poles. This time we use a different expansion.

Example:

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Positive powers of z.

$$X(z) = \frac{1}{(z+1)(z-1)^2}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

Determine the coefficients A_1, A_2, and A_3.

$$A_1 = \frac{(z+1)X(z)}{z}\Big|_{z=-1} = \frac{z^2}{(z-1)^2}\Big|_{z=-1} = \frac{1}{4}$$

$$A_3 = \frac{(z-1)^2 X(z)}{z}\Big|_{z=1} = \frac{z^2}{(z+1)}\Big|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left[\frac{(z-1)^2 X(z)}{z}\right]_{z=1} = \frac{d}{dz}\left[\frac{z^2}{(z+1)}\right]_{z=1} = \frac{2z(z+1)-z^2}{(z+1)^2}\Big|_{z=1} = \frac{2(2)-1}{2^2} = \frac{3}{4}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{\frac{1}{4}}{(z+1)} + \frac{\frac{3}{4}}{(z-1)} + \frac{\frac{1}{2}}{(z-1)^2}$$

$$X(z)=\frac{1}{4}\frac{z}{(z+1)}+\frac{3}{4}\frac{z}{(z-1)}+\frac{1}{2}\frac{z}{(z-1)^2}$$

$$X(z)=\frac{1}{4}\frac{1}{(1+z^{-1})}+\frac{3}{4}\frac{1}{(1-z^{-1})}+\frac{1}{2}\frac{z^{-1}}{(1-z^{-1})^2}$$

$$X(z)=\left[\frac{1}{4}(-1)^n+\frac{3}{4}+\frac{1}{2}n\right]u(n)$$

<u>Distinct complex poles</u>

<u>Example:</u>

$$X(z)=\frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Positive powers of z.

$$X(z)=\frac{z(z+1)}{z^2-z+0.5}$$

$$\frac{X(z)}{z}=\frac{z+1}{z^2-z+0.5}=\frac{z+1}{(z-p_1)(z-p_2)}=\frac{A_1}{(z-\frac{1}{2}-j\frac{1}{2})}+\frac{A_2}{(z-\frac{1}{2}+j\frac{1}{2})}$$

$$A_1=\frac{(z-p_1)X(z)}{z}\bigg|_{z=p_1}=\frac{z+1}{(z-\frac{1}{2}+j\frac{1}{2})}\bigg|_{z=\frac{1}{2}+j\frac{1}{2}}=\frac{\frac{1}{2}+j\frac{1}{2}+1}{(\frac{1}{2}+j\frac{1}{2}-\frac{1}{2}+j\frac{1}{2})}=\frac{1}{2}-j\frac{3}{2}$$

$$A_2=\frac{(z-p_2)X(z)}{z}\bigg|_{z=p_2}=\frac{z+1}{(z-\frac{1}{2}-j\frac{1}{2})}\bigg|_{z=\frac{1}{2}-j\frac{1}{2}}=\frac{\frac{1}{2}-j\frac{1}{2}+1}{(\frac{1}{2}-j\frac{1}{2}-\frac{1}{2}-j\frac{1}{2})}=\frac{1}{2}+j\frac{3}{2}$$

$$\frac{X(z)}{z}=\frac{z+1}{z^2-z+0.5}=\frac{\frac{1}{2}-j\frac{3}{2}}{(z-\frac{1}{2}-j\frac{1}{2})}+\frac{\frac{1}{2}+j\frac{3}{2}}{(z-\frac{1}{2}+j\frac{1}{2})}$$

$$A_1=\frac{1}{2}-j\frac{3}{2}=\frac{\sqrt{10}}{2}e^{-j71.565} \qquad A_2=\frac{1}{2}+j\frac{3}{2}=\frac{\sqrt{10}}{2}e^{+j71.565}$$

$$A_1=A_2^* \qquad p_1=\frac{1}{2}+j\frac{1}{2}=\frac{1}{\sqrt{2}}e^{+jp/4}, \;\; p_1=p_2^*$$

$$\frac{X(z)}{z}=\frac{z+1}{z^2-z+0.5}=\frac{\frac{\sqrt{10}}{2}e^{-j71.565}}{(z-\frac{1}{\sqrt{2}}e^{jp/4})}+\frac{\frac{\sqrt{10}}{2}e^{j71.565}}{(z-\frac{1}{\sqrt{2}}e^{-jp/4})}$$

$$x(n)=\frac{\sqrt{10}}{2}e^{-j71.565}\left(\frac{1}{\sqrt{2}}e^{-jp/4}\right)^n u(n)+\frac{\sqrt{10}}{2}e^{j71.565}\left(\frac{1}{\sqrt{2}}e^{-jp/4}\right)^n u(n)$$

$$x(n)=\sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n\left(\frac{1}{2}e^{j\left(\frac{p\,n}{4}-71.565\right)}+\frac{1}{2}e^{-j\left(\frac{p\,n}{4}-71.565\right)}\right)u(n)$$

$$x(n)=\sqrt{10}\left(\frac{1}{\sqrt{2}}\right)^n\cos\left(\frac{p\,n}{4}-71.565\right)u(n)$$

The One-Side z-Ttransform

The one-sided z-transform of a signal x(n) is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$x(n) \overset{z^+}{\leftrightarrow} X^+(z)$$

The one-sided z-transform has the following characteristics:

1. It does not contain information about the signal x(n) for negative values of time (i.e., for n<0)
2. It is unique only for causal signals, because only these signals are zero for n<0.
3. The one-sided z-transform $X^+(z)$ of x(n) is edantical to the two-sided z-transform of the signal x(n)u(n).
4. ROC of $X^+(z)$ is always the exterior of the circle. So it is not necessary to refer to their ROC.

Example:

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\} \overset{x^+}{\leftrightarrow} X_1^+(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\} \overset{z^+}{\leftrightarrow} X_2^+(z) = 5 + 7z^{-1} + z^{-3}$$

$$x_3(n) = \{\underset{\uparrow}{0}, 0, 1, 2, 5, 7, 0, 1\} \overset{x^+}{\leftrightarrow} X_3^+(z) = 1z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

$$x_4(n) = d(n) \overset{x^+}{\leftrightarrow} X_4^+(z) = 1, \quad x_5(n) = d(n-k) \overset{x^+}{\leftrightarrow} X_5^+(z) = z^{-k}$$

$$x_6(n) = d(n+k) \overset{x^+}{\leftrightarrow} X_6^+(z) = 0$$

$$x_4(n) = a^n u(n) \overset{x^+}{\leftrightarrow} X_4^+(z) = \frac{1}{1 + az^{-1}}$$

Shifting Property

Case 1:Time Delay

If $x(n) \overset{x^+}{\leftrightarrow} X^+(z)$

Then

$$x(n-k) \overset{x^+}{\leftrightarrow} z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n)z^{n}\right] \quad k>0$$

In case x(n) is causal, then

$$x(n-k) \overset{x^+}{\leftrightarrow} z^{-k} X^+(z) \quad k>0$$

Example:

$$x(n) = a^n u(n) \overset{x^+}{\leftrightarrow} X^+(z) = \frac{1}{1+az^{-1}}$$

The z-transform of $x_1(n) = x(n-2)$

$$x(n-2) = a^{n-2}u(n-2) \overset{x^+}{\leftrightarrow} z^{-2}\left[\frac{1}{1+az^{-1}} + x(-1)z + x(-2)z^2\right]$$

$$= \frac{z^{-2}}{1+az^{-1}} + x(-1)z^{-1} + x(-1)$$

$$= \frac{z^{-2}}{1+az^{-1}} + a^{-1}z^{-1} + a^{-2}$$

Case 2: Time advance

If $x(n) \overset{x^+}{\leftrightarrow} X^+(z)$

Then

$$x(n+k) \overset{x^+}{\leftrightarrow} z^{k}\left[X^+(z) - \sum_{n=0}^{k-1} x(-n)z^{-n}\right] \quad k>0$$

Example:

$$x(n) = a^n u(n) \overset{x^+}{\leftrightarrow} X^+(z) = \frac{1}{1+az^{-1}}$$

The z-transform of $x_1(n) = x(n+2)$

$$x(n-2) = a^{n-2}u(n-2) \overset{x^+}{\leftrightarrow} z^2\left[\frac{1}{1+az^{-1}} + x(0) + x(1)z^{-1}\right]$$

$$= \frac{z^2}{1+az^{-1}} + x(0)\,z^2 + x(1)\,z^1$$

$$= \frac{z^2}{1+az^{-1}} + z^2 + az^1$$

Final Value Theorem:

If $x(n) \overset{x^+}{\leftrightarrow} X^+(z)$

Then

$$\lim_{n\to\infty} x(n) \overset{x^+}{\leftrightarrow} \lim_{z\to 1}(z-1)X^+(z)$$

Thisis exits if the ROCof $(z-1)X^+(z)$includes the unit circle.

Example:

$x(n) = u(n)$, $h(n) = a^n u(n)$

$y(n) = h(n) * x(n)$

$$Y(z) = H(z)X(z) = \frac{1}{1-az^{-1}}\frac{1}{1-z^{-1}} = \frac{z^2}{(z-a)(z-1)}$$ ROC: $|z| > |a|$

$$(z-1)Y(z) = (z-1)\frac{z^2}{(z-a)(z-1)} = \frac{z^2}{(z-a)}$$ ROC: $|z| > |a|$

$$\lim_{n\to\infty} x(n) \overset{x^+}{\leftrightarrow} \lim_{z\to 1}\frac{z^2}{z-a} = \frac{1}{1-a}$$

Solution of Difference Equation:

The one-sided z-transform is a very efficient tool for the solution of difference equation.

Example: Let y(n0 represents the Fibonnacci series

$y(n) = y(n-1) + y(n-2)$

The initial condition

$y(0) = y(-1) + y(-2) = 1$

$y(1) = y(0) + y(-1) = 1$ → $y(-1) = 0$, $y(-2) = 1$

$$Y(z) = z^{-1}\left[Y^{+}(Z) + y(-1)\right] + z^{-2}\left[Y^{+}(z) + y(-1) \_ y(-1)z^{-1}\right]$$

$$Y(z) = \frac{1}{1 - z^{-1} + z^{-2}} = \frac{z^2}{z^2 - z + 1}$$

$$\frac{Y(z)}{z} = \frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)\left(z - \frac{1-\sqrt{5}}{2}\right)} = \frac{A_1}{\left(z - \frac{1+\sqrt{5}}{2}\right)} + \frac{A_2}{\left(z - \frac{1-\sqrt{5}}{2}\right)}$$

$$A_1 = \left.\frac{z}{\left(z - \frac{1-\sqrt{5}}{2}\right)}\right|_{z=\frac{1+\sqrt{5}}{2}} = \frac{\frac{1+\sqrt{5}}{2}}{\sqrt{5}}$$

$$A_2 = \left.\frac{z}{\left(z - \frac{1+\sqrt{5}}{2}\right)}\right|_{z=\frac{1-\sqrt{5}}{2}} = \frac{-\left(\frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}}$$

$$y(n) = \left[\frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1-\sqrt{5}}{2\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]u(n)$$