The inverse of the z-transform

The inverse z-transform is formally given by

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

There are three methods for evaluation of inverse z-transform

1. Direct evaluation by Contour integration (using Cauchy residue theorem)
2. Expansion into a series of term in the variable z and z^{-1}
3. Partial-fraction expansion and table lookup.

2. The Inverse z-transform by Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$ with given ROC.

Example:
Using Power Series Expansion to determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

a. ROC: $|z| > 1$
b. ROC: $|z| < 0.5$

Solution:

a. Since the ROC is the exterior of the circle, we can expect x(n) to be a causal.

$$
\begin{array}{r|l}
 & 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\ldots \\
\hline
1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} & 1
\end{array}
$$

$$1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1}-\frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3}$$

$$\frac{7}{4}z^{-2}-\frac{3}{4}z^{-3}$$

$$\frac{7}{4}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4}$$

$$\frac{15}{8}z^{-3}-\frac{7}{8}z^{-4}$$

$$\frac{15}{8}z^{-3}-\frac{45}{16}z^{-4}+\frac{15}{16}z^{-5}$$

$$\frac{31}{16}z^{-4}-\frac{15}{16}z^{-5}$$

Therefore,

$$X(z)=1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+.....$$

x(n) con be obtained as

$$x(n)=\{\underset{\uparrow}{1},\frac{3}{2},\frac{7}{4},\frac{15}{8},\frac{31}{16},.....\}$$

Using MatLab

$$X(z)=\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

```
>> y=[1];                              //y={1}
>> h=[1 –1.5 0.5];                     //h={1, -1.5, 0.5}
>> n=7;
>> y=[y zeros(1,n-1)];                 //y={1,0,0,0,0,0,0}
>> [x,r]=deconv(y,h)
x =
  1.0000   1.5000   1.7500   1.8750   1.9375   1.9688
r =
    0        0        0        0        0        0     1.9844   -0.9844
```

Therefore

$$x(0) = 1, \; x(1) = 1.5, \; x(2) = 1.75, \; x(3) = 1.875, \; x(4) = 1.9375$$

b. ROC is the interior of a circle. The signal x(n) is anticausal. The long division in the following way.

$$\begin{array}{r l}
 & 2z^2 + 6z^3 + 4z^4 + 30z^5 + \ldots \\
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 \,\big) & 1 \\
 & 1 - 3z + 2z^2 \\ \hline
 & 3z - 2z^2 \\
 & 3z - 9z^2 + 6z^3 \\ \hline
 & 7z^2 - 6z^3 \\
 & 7z^2 - 21z^3 + 14z^4 \\ \hline
 & 15z^3 - 14z^4 \\
 & 15z^3 - 45z^4 + 30z^5 \\ \hline
 & 31z^4 - 30z^5
\end{array}$$

Therefore,

$$X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + \ldots..$$

x(n) can be obtained as

$$x(n) = \{\ldots 30, 14, 6, 2, 0, 0\}$$

$$\uparrow$$

The inverse z-transform by Partial Fraction Expansion

The function X(z) as a linear combination of

$$X(z) = a_1 X_1(z) + a_2 X_2(z) + \ldots + a_K X_K(z)$$

If we know the inverse transform of each $X_i(z)$ from the table. Then we can write

$$x(n) = a_1 x_1(n) + a_2 x_2(n) + \ldots + a_K x_K(n)$$

we usually have X(z) function in a rational function form.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$

A rational function is called proper if $M < N$.
A rational function is called improper if $M \geq N$.
An improper rational function ($M \geq N$) can be written as the sum of polynomial and a proper rational function.

$$X(z) = c_0 + c_1 z^{-1} + \ldots + c_{M-n} z^{-(M-N)} + \frac{N_1(z)}{D(z)}$$

Example:

$$X(z) = \frac{1 + 3z^{-1} + \frac{11}{6} z^{-2} + \frac{1}{3} z^{-3}}{1 + \frac{5}{6} z^{-1} + \frac{1}{6} z^{-2}}$$

$$
\begin{array}{r|l}
 & 2z^{-1}+1 \\
\hline
\frac{1}{6}z^{-2}-\frac{5}{6}z^{-1}+1 & \frac{1}{3}z^{-3}+\frac{11}{6}z^{-2}+3z^{-1}+1 \\
 & \underline{\frac{1}{3}z^{-3}-\frac{5}{3}z^{-2}+2z^{-1}} \\
 & \qquad \frac{1}{6}z^{-2}-z^{-1}+1 \\
 & \qquad \underline{\frac{1}{6}z^{-2}-\frac{5}{6}z^{-1}+1} \\
 & \qquad\qquad \frac{1}{6}z^{-1}
\end{array}
$$

$$X(z)=1+2z^{-1}\frac{\frac{1}{6}z^{-1}}{1+\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

For proper rational function

$$X(z)=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}$$

Step.1: Eliminate negative powers of z, multiply by z^N

$$X(z)=\frac{b_0z^N+b_1z^{N-1}+\ldots+b_Mz^{N-M}}{a_0z^N+a_1z^{N-1}+\ldots+a_Nz}$$

Step.2: Make proper rational function divided the both side to z

$$\frac{X(z)}{z}=\frac{b_0z^{N-1}+b_1z^{N-2}+\ldots+b_Mz^{N-M-1}}{a_0z^N+a_1z^{N-1}+\ldots+a_Nz}$$

Step.3: Find the poles (denominator roots). We can have distinct real poles, distinct complex poles, multiple-order real poles, and multiple-order complex poles.

A. <u>Distinct Poles</u>: The poles are all different

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \ldots + \frac{A_N}{z - p_N}$$

We can determine $A_1, A_2, \ldots A_N$ using two different methods.
Let's give an example and use the both methods.

Example:
Determine the partial fraction expansion the following function

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Step.1 $$X(z) = \frac{z^2}{z^2 - 1.5z + 0.5}$$

Step.1 $$\frac{X(z)}{z} = \frac{z}{z^2 - 1.5z + 0.5}$$

Step.3 $$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

<u>Method.1</u>

$$A_1 = \frac{(z-1)X(z)}{z}\Big|_{z=1} = \frac{z}{(z-0.5)}\Big|_{z=1} = \frac{1}{1-0.5} = 2$$

$$A_2 = \frac{(z-0.5)X(z)}{z}\Big|_{z=1} = \frac{z}{(z-1)}\Big|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

Method.2

Multiply the denominator one with (z-0.5) and the denominator two with (x-1) to make them equal. Then we have

$$z = (z-0.5)A_1 + (z-1)A_2$$

If z=1 → $z = (z-0.5)A_1\big|_{z=1}$ → $1 = (1-0.5)A_1$, → $A_1 = 2$

If z=0.5 → $z = (z-1)A_2\big|_{z=0.5}$ → $0.5 = (0.5-1)A_2$, → $A_2 = -1$

Finally, we have

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

Multiply to z the both side

$$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

Divide by z the right side

$$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

From the z-transform table

$$x(n) = \left[2-(0.5)^n\right]u(n)$$

In general form:

$$Z^{-1}\left\{\frac{1}{1-p_k z^{-1}}\right\} = \begin{cases} p_k^n u(n) & \text{if } ROC: |z| > |p_k| \text{ (causal signal)} \\ -(p_k^n)u(n) & \text{if } ROC: |z| < |p_k| \text{ (anticausal signal)} \end{cases}$$

Using MatLab to find inverse z-transform:

```
>> syms z n
>> iztrans(1/(1-1.5*z^-1+0.5*z^-2))
 ans =
 2-(1/2)^n
```