February 18,2002
ECE308- 03 # 7.1

3.3. Rational Z-Transform

Poles and Zeros

The *poles* of a z-transform are the values of z for which if $X(z)=\infty$

The *zeros* of a z-transform are the values of z for which if $X(z)=0$

$X(z)$ is in rational function form

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0+b_1z^{-1}+\ldots+b_Mz^{-M}}{a_0+a_1z^{-1}+\ldots+a_Nz^{-N}}=\frac{\sum_{k=0}^{M}b_kz^{-k}}{\sum_{k=0}^{M}a_kz^{-k}}$$

If $a_0\neq 0$ and $b_0\neq 0$

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0z^{-M}}{a_0z^{-N}}\frac{\left(z^M+\frac{b_1}{b_0}z^{M-1}+\ldots+\frac{b_M}{b_0}\right)}{\left(z^N+\frac{a_1}{a_0}z^{N-1}+\ldots+\frac{a_N}{a_0}\right)}$$

$$X(z)=\frac{N(z)}{D(z)}=\frac{b_0}{a_0}z^{-M+N}\frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}$$

$$X(z)=\frac{N(z)}{D(z)}=Gz^{N-M}\frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

M finite zeros at $z=z_1,z_2,\ldots,z_M$

N finite poles at $z = p_1, p_2, ..., p_N$

And $|N - M|$ zeros if $N > M$ or poles if $M > N$ at the origin $z = 0$

A zero exists at $z = \infty$ if $Z(\infty) = 0$ and a pole exists at $z = \infty$ if

We can represent X(z) graphically by a pole-zero plot in complex plane.

Shows the location of poles by (x)

Shows the location of zeros by (o).

Definition of ROC of a z-transform should not contain any poles.

Example:

Determine the pole-zero plot for the signal

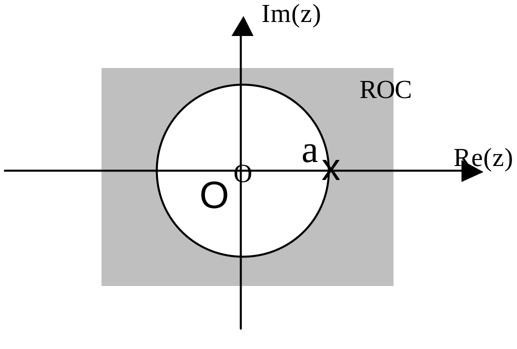
$$x(n) = a^n u(n) \; a > 0$$

The z-transform is

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

One zero at $z_1 = 0$

One pole at $p_1 = a$. $p_1 = a$ is not included in the ROC.



Example:

Determine the pole-zero plot for the signal

$$x(n) = \begin{cases} a^n & 0 \le n \le M - 1 \\ 0 & elsewhere \end{cases} \quad \text{and } a > 0$$

The z-transform is

$$X(z) = \sum_{n=0}^{M-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}\left(z-a\right)}$$

$$X(z) = \frac{\left(z - z_1\right)\left(z - z_2\right)...\left(z - z_{M-1}\right)}{z^{M-1}}$$

It has M-1 zeros and M-1 poles.

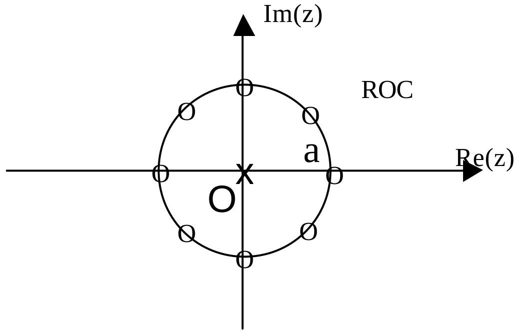
For M=8

The zeros $z_k = ae^{j2p\,k/M}$

$z_1 = ae^{j2p/8}, z_2 = ae^{j4p/8}$. … $z_7 = ae^{j14p/8}$

the poles at $p = 0$

ROC:the entire z-plane except $z = 0$.



Pole Location and Time-Domain behavior for Causal Signals

For the signal

$$x(n) = a^n u(n)\ a > 0$$

The z-transform is

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

One zero at $z_1 = 0$. One pole at $p_1 = a$.

The signal is decaying $0 < a < 1$
The signal is fixed if $a = 1$
The signal is growing if $a > 1$
The signal alternates if a is negative

Causal signals with poles outside the unit circle become unbounded.

For the signal

$$x(n) = na^n u(n) \; a > 0$$

The z-transform is

$$X(z) = \frac{az^{-1}}{\left(1 - az^{-1}\right)^2} = \frac{za}{\left(z - a\right)^2}$$

One zero at $z_1 = 0$. Two pole at $p_1 = a$.
Causal signals with poles outside and equal the unit circle become unbounded.

The system Function of a Linear Time –Invariant System

The input sequence $x(n)$
The output sequence $y(n)$
The relationship in the z-domain

$$Y(z) = H(z)\,X(z)$$

H(z) is obtained

$H(z) = \dfrac{Y(z)}{X(z)}$ is called *system function*

We know that

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

We can take the inverse of z-transform to find h(n).

H(z) can be obtained from a linear constant-coefficien difference equation.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{N} b_k x(n-k)$$

The z-transform

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{N} b_k X(z) z^{-k}$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = \sum_{k=0}^{N} b_k X(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right)}$$

Example:

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

The z-transform from the difference equation

$$Y(z) = \frac{1}{2} Y(z) z^{-1} + 2X(z)$$

The system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

The suystem has one pole at $z = 1/2$ and one zero at $z = 0$

The inverse z-transform

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

$$x_2(n) = 3^n u(n) \overset{z}{\leftrightarrow} X_2(z)\frac{1}{1-3z^{-1}} \quad \text{ROC: } |z| > 3$$