Properties of z-Transform

Linearity:

If $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

Then $x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z)$

Example:

$$x(n) = \left[3(2^n) - 4(3^n)\right]u(n)$$

$$x_1(n) = 2^n u(n)$$

$$x_2(n) = 3^n u(n)$$

$$x(n) = 3x_1(n) - 4x_2(n)$$

Its z-transform:

$$X(z) = 3X_1(z) - 4X_2(z)$$

$$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - a z^{-1}} \quad \text{ROC: } |z| > |a|$$

$$x_1(n) = 2^n u(n) \overset{z}{\leftrightarrow} X_1(z) \frac{1}{1 - 2z^{-1}} \quad \text{ROC: } |z| > 2$$

$$x_2(n) = 3^n u(n) \overset{z}{\leftrightarrow} X_2(z) \frac{1}{1 - 3z^{-1}} \quad \text{ROC: } |z| > 3$$

The intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$

$$X(z) = \frac{2}{1 - 2z^{-1}} - \frac{4}{1 - 3z^{-1}} \quad \text{ROC: } |z| > 3$$

Example: Find z-transform of the following function

$$x(n) = (\cos w_0 n) u(n)$$

Using Euler's identity:

$$x(n) = (\cos w_0 n) u(n) = \frac{1}{2} e^{jw_0 n} u(n) + \frac{1}{2} e^{-jw_0 n} u(n)$$

$$X(z) = \frac{1}{2} Z\left\{e^{jw_0 n} u(n)\right\} + \frac{1}{2} Z\left\{e^{-jw_0 n} u(n)\right\}$$

We set

$$a = e^{\pm jw_0} \quad (|a| = \left|e^{\pm jw_0}\right| = 1$$

$$e^{jw_0 n} u(n) \leftrightarrow \frac{1}{1 - e^{jw_0} z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$e^{-jw_0 n} u(n) \leftrightarrow \frac{1}{1 - e^{-jw_0} z^{-1}}, \quad \text{ROC: } |z| > 1$$

$$X(z) = \frac{1}{2} \frac{1}{1 - e^{jw_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-jw_0} z^{-1}}, \quad \text{ROC: } |z| > 1$$

Some algebraic manipulation:

$$(\cos w_0 n) u(n) \overset{z}{\leftrightarrow} \frac{1 - z^{-1} \cos w_0}{1 - 2z^{-1} \cos w_0 + z^{-2}} \quad \text{ROC: } |z| > 1$$

For sin signal:

$$(\sin w_0 n) u(n) \overset{z}{\leftrightarrow} \frac{z^{-1} \sin w_0}{1 - 2z^{-1} \cos w_0 + z^{-2}} \quad \text{ROC: } |z| > 1$$

Time Shifting

If $x(n) \overset{z}{\leftrightarrow} X(z)$

Then $x(n-k) \overset{z}{\leftrightarrow} z^{-k}X(z)$

ROC: same as X(z) accept for $z=0$ if $k>0$ and $z=0$ if $k<0$

Example:

$$x_1(n)=\{1,2,5,7,0,1\}$$
$$\uparrow$$

$$X_1(z)=1+2z^{-1}+5z^{-2}+7z^{-3}+z^{-5}, \text{ ROC: } z \neq 0$$

$$x_2(n)=x_1(n+2)$$
$$X_2(z)=z^2+2z^1+5+7z^{-1}+z^{-3}, \text{ ROC: } z\neq 0, z\neq\infty$$

$$x_2(n)=x_1(n-2)$$
$$X_3(z)=z^{-2}+2z^{-3}+5z^{-4}+7z^{-5}+z^{-7}, \text{ ROC: } z\neq 0$$

Scaling in the z-domain

If $x(n) \overset{z}{\leftrightarrow} X(z)$ ROC: $r_1<|z|<r_2$

then $a^n x(n) \overset{z}{\leftrightarrow} X(a^{-1}z)$ ROC: $|a|r_1<|z|<|a|r_2$

for any constant a, real or complex.

Example:

$$x(n)=a^n(\cos w_0 n)u(n)$$

$$(\cos w_0 n)u(n) \overset{z}{\leftrightarrow} \frac{1-z^{-1}\cos w_0}{1-2z^{-1}\cos w_0+z^{-2}} \text{ ROC: } |z|>1$$

$$X(z) = \frac{1 - az^{-1}\cos w_0}{1 - 2az^{-1}\cos w_0 + a^2 z^{-2}} \qquad \text{ROC: } |z| > |a|$$

Time reversal

If $x(n) \overset{z}{\leftrightarrow} X(z)$ ROC: $r_1 < |z| < r_2$

then $x(-n) \overset{z}{\leftrightarrow} X(z^{-1})$ ROC: $\frac{1}{r_1} < |z| < \frac{1}{r_2}$

Example:

$$x(n) = u(-n)$$

$$u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - z^{-1}} \quad \text{ROC: } |z| > 1$$

$$u(-n) \overset{z}{\leftrightarrow} \frac{1}{1 - z} \quad \text{ROC: } |z| < 1$$

Differentiation in the z-domain

If $x(n) \overset{z}{\leftrightarrow} X(z)$ ROC: $r_1 < |z| < r_2$

then $nx(n) \overset{z}{\leftrightarrow} -z\frac{dX(z)}{dz}$ ROC: $r_1 < |z| < r_2$

Example:

$$x(n) = na^n u(n)$$

$$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - az^{-1}} \quad \text{ROC: } |z| > |a|$$

$$na^n u(n) \overset{z}{\leftrightarrow} X(z) = -z\frac{d}{dz}\left(\frac{1}{1 - az^{-1}}\right) = \frac{az^{-1}}{(1 - az^{-1})^2}$$

ROC: $|z| > |a|$

Convolution of two sequences

If $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$

$x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

then $x(n) = x_1(n) * x_2(n) \overset{z}{\leftrightarrow} X(z) = X_1(z)X_2(z)$

ROC: is the intersection of $X_1(z)$ and $X_2(z)$

Example:

$$x_1(n) = \{1, -2, 1\}$$

$$x_2(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{elsewhere} \end{cases}$$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X(z) = X_1(z)X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Therefore,

$$x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$$

Correlation of two sequences:

If $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$

$x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

then

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \overset{z}{\leftrightarrow} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$$

Example:

The autocorrelation of signal

$x(n) = a^n u(n) \quad -1 < a < 1$

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{ROC: } |z| > |a|$$

$$X(z^{-1}) = \frac{1}{1 - az}, \qquad \text{ROC: } |z| > \frac{1}{|a|}$$

$$R_{x_1x_1}(z) = X_1(z)X_1(z^{-1})\, R_{x_1x_1}(z) = \frac{1}{1 - az^{-1}}\frac{1}{1 - az} = \frac{1}{1 - a(z + z^{-1}) + a^2}$$

The initial Value Theorem:

If $x(n)$ is causal ($x(n) = 0$ for $n > 0$)

$$x(0) = \lim_{z \to \infty} X(z)$$