Z Transform and Its Application to the Analysis of LTI Systems

- Z-transform is an alternative representation of a discrete signal.
- Z-Transform is important in the analysis and characterization of LTI systems
- Z-Transform play the same role in the analysis of discrete time signal and LTI systems as Laplace transform does in the analysis of continuous time signal and LTI systems.
- Z-transform provides us with a mean of characterizing an LTI system and its response to various signals by its pole-zero locations.

The direct Z-Transform:

- The z-transform of a discrete time signal x(n0 is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

←The direct z-transform

where z is complex variable.
The inverse procedure is called *inverse z-transform*. So relationship can give as

$$x(n) \overset{z}{\leftrightarrow} X(z)$$

We also denote the z-transform as

$$X(z) \equiv Z[x(n)]$$

Since the z-transform is an infinite power series, it is exists only for the series convergence. The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value.

Example:

$$x(n) = \{1, \underset{\uparrow}{2}, 5, 7, 0, 1\}$$

$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}, \text{ ROC: } z \neq 0$$

$z^{k} (k > 0)$ becomes unbounded for $z = \infty$

$z^{-k} (k > 0)$ becomes unbounded for $z = 0$

In many cases , we can express the sum of the finite or infinite series for z-transform in a closed-form expression.

Example:

$$x(n) = \left(\frac{1}{2}\right)^{n} u(n)$$

$$x(n) = \left\{1, \frac{1}{2}, \left(\frac{1}{2}\right)^{2}, \left(\frac{1}{2}\right)^{3}, \left(\frac{1}{2}\right)^{4}, \ldots\right\}$$

$$X(z) = 1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^{2} z^{-2} + \left(\frac{1}{2}\right)^{3} z^{-3} + \left(\frac{1}{2}\right)^{4} z^{-4} + \ldots$$

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^{n}$$

If $\left|\frac{1}{2} z^{-1}\right| < 1$ or $|z| > \frac{1}{2}$

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} \qquad \text{ROC: } |z| > \frac{1}{2}$$

Remind:

$$1 + A + A^2 + A^3 + ... = \frac{1}{1 - A} \ \ if \ |A| < 1$$

The complex variable z can be written in polar form as

$$z = re^{jq}$$

The X(z) can be expressed as

$$X(z)\Big|_{z=re^{jq}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-jqn}$$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-jqn} \right| \leq \sum_{n=-\infty}^{\infty} \left| x(n) r^{-n} \right|$$

$$\leq \sum_{n=-\infty}^{-1} \left| x(n) r^{-n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| = \sum_{n=1}^{\infty} \left| x(-n) r^n \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

In the first sum, there must be exist values of r small enough so that $x(-n)r^n$ is absolutely summable.

In the second sum, there must exist values of r large enough so that $x(n)/r^n$ is absolutely summable.

Since the convergence of X(z) requires that both sums, the ROC of X(z) is specified the common region where the both sum are finite.

If there is no common region , then X(z) does not exist.

Example:

$$x(n) = a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

if $\left| a z^{-1} \right| < 1$ the power series is

$$X(z) = \frac{1}{1 - a z^{-1}} \quad \text{ROC: } |z| > |a|$$

Example:

$$x(n) = -a^n u(-n-1) = \begin{cases} 0 & n \geq 0 \\ -a^n & n \leq -1 \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{-1} (-a)^n z^{-n} = -\sum_{m=1}^{\infty} (a^{-1} z)^m$$

if $\left| a^{-1} z \right| < 1$ the power series is

$$X(z) = -\frac{a^{-1} z}{1 - a^{-1} z} = \frac{1}{1 - a z^{-1}} \quad \text{ROC: } |z| < |a|$$

The both signal has same z-transform.

$$Z\left\{a^n u(n)\right\} = Z\left\{-a^n u(-n-1)\right\} = \frac{1}{1 - a z^{-1}}$$

Example:

$$x(n) = a^n u(n) + b^n u(-n-1)$$

$$X(z) = \sum_{n=0}^{\infty} (a)^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n + \sum_{m=1}^{\infty} (b^{-1} z)^m$$

From the first power series if $\left| a z^{-1} \right| < 1$ or $|z| > |a|$ and from the second power series if $\left| b^{-1} z \right| < 1$or $|z| < |b|$, then

We have two cases:

a. $|b| < |a|$: There is no common region, X(z) does not exist.

b. $|b| > |a|$: There is common region, which is $|a| < |z| < |a|$,

Then we obtain

$$X(z) = \frac{1}{1 - a z^{-1}} - \frac{1}{1 - b^{-1} z} = \frac{b - a}{a + b - z - a b z^{-1}} \quad \text{ROC: } |a| < |z| < |b|$$

The Inverse z-Transform:

X(n) can be obtain from X(z) using the Cauchy integral theorem. Multiplying both side z^{n-1} and integral both sides over a closed contour with in ROC of X(z)which encloses the origin.

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{k=-\infty}^{\infty} x(k) z^{n-1-k} dz$$

$$\oint_C X(z) z^{n-1} dz = \sum_{k=-\infty}^{\infty} x(k) \oint_C z^{n-1-k} dz$$

The Cauchy integral theorem, which states thet

$$\frac{1}{2p\, j} \oint_C z^{n-1-k} dz = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

Applying this the previous equation, we have inversion formula

$$x(n) = \frac{1}{2p\, j} \oint_C X(z) z^{n-1} dz$$

Properties of z-Transform

<u>Linearity:</u>

If $x_1(n) \overset{z}{\leftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\leftrightarrow} X_2(z)$

Then $x(n) = a_1 x_1(n) + a_2 x_2(n) \overset{z}{\leftrightarrow} a_1 X_1(z) + a_2 X_2(z)$

Example:

$$x(n) = \left[3(2^n) - 4(3^n)\right] u(n)$$

$$x_1(n) = 2^n u(n)$$

$$x_2(n) = 3^n u(n)$$

$$x(n) = 3x_1(n) - 4x_2(n)$$

Its z-transform:

$$X(z) = 3X_1(z) - 4X_2(z)$$

$a^n u(n) \overset{z}{\leftrightarrow} \frac{1}{1 - a z^{-1}}$ ROC: $|z| > |a|$

$x_1(n) = 2^n u(n) \overset{z}{\leftrightarrow} X_1(z) \frac{1}{1 - 2z^{-1}}$ ROC: $|z| > 2$

$x_2(n) = 3^n u(n) \overset{z}{\leftrightarrow} X_2(z) \frac{1}{1 - 3z^{-1}}$ ROC: $|z| > 3$

The intersection of the ROC of $X_1(z)$ and $X_2(z)$ is $|z| > 3$

$X(z) = \frac{2}{1 - 2z^{-1}} - \frac{4}{1 - 3z^{-1}}$ ROC: $|z| > 3$

Example: Find z-transform of the following function

$$x(n) = (\cos w_0 n) u(n)$$

Using Euler's identity:

$$x(n) = (\cos w_0 n) u(n) = \frac{1}{2} e^{jw_0 n} u(n) + \frac{1}{2} e^{-jw_0 n} u(n)$$

$$X(z) = \frac{1}{2} Z\left\{ e^{jw_0 n} u(n) \right\} + \frac{1}{2} Z\left\{ e^{-jw_0 n} u(n) \right\}$$

We set

$$a = e^{\pm jw_0 n} \quad (|a| = |e^{\pm jw_0 n}| = 1$$

$$e^{jw_0 n} u(n) \leftrightarrow \frac{1}{1 - e^{jw_0} z^{-1}}, \quad \text{ROC:} |z| > 1$$

$$e^{-jw_0 n} u(n) \leftrightarrow \frac{1}{1 - e^{-jw_0} z^{-1}}, \quad \text{ROC:} |z| > 1$$

$$X(z) = \frac{1}{2} \frac{1}{1 - e^{jw_0} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-jw_0} z^{-1}}, \quad \text{ROC:} |z| > 1$$

Some algebraic manipulation:

$$(\cos w_0 n) u(n) \overset{z}{\leftrightarrow} \frac{1 - z^{-1} \cos w_0}{1 - 2z^{-1} \cos w_0 + z^{-2}} \quad \text{ROC:} |z| > 1$$

For Sin signal:

$$(\sin w_0 n) u(n) \overset{z}{\leftrightarrow} \frac{z^{-1} \sin w_0}{1 - 2z^{-1} \cos w_0 + z^{-2}} \quad \text{ROC:} |z| > 1$$