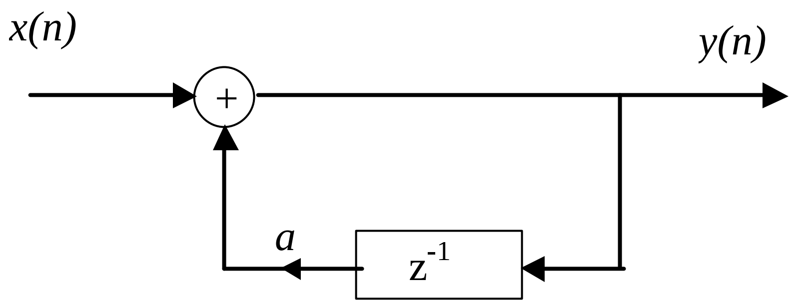
2.4.2 Constant-Coefficient Difference Equations

LTI Systems can be described by Constant-Coefficient Difference Equations to represent the input-output relations.

Let's have a recursive system that is first-order difference



$$y(n) = ay(n-1) + x(n)$$

where a is a constant and system is time invariant. We assume that we have initial condition y(-1).

For $n \geq 0$, y(n) can be obtained

$$y(0) = ay(-1) + x(0)$$

$$y(1) = ay(0) + x(1) = a^2 y(-1) + ax(0) + x(1)$$

$$y(2) = ay(1) + x(2) = a^3 y(-1) + a^2 x(0) + ax(1) + x(2)$$

…. ….

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + ... + x(n) \qquad \text{or}$$

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k) \qquad \text{, for } n \geq 0$$

The response y(n) of the system depends on

- initial condition y(-1) of the system and
- the system response to the input signal.

If the system is initially relaxed at time n=0, its memory should be zero. So, y(-1)=0.

Then, system is at zero state and the corresponding output is called zero-state response or *forced response*.

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k) \ , \quad \text{for } n \geq 0$$

If system is initially nonrelaxed (y(-1)≠0) and the input x(n)=0 for all n.

The corresponding output is called *zero-input response* or *natural response*.

$$y_{zi}(n) = a^{n+1} y(-1) \qquad , \text{for } n \geq 0$$

The total system response is

$$y(n) = y_{zs}(n) + y_{zi}(n) \quad , \text{for } n \geq 0$$

General form of linear constant-coefficient difference equation is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \text{or}$$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

N is called the *order of the difference equation* or *order of the system.*

In order to find y(n), we need to know initial conditions y(n-1), y(-2),…,y(n-N) and the input x(n) for all $n \geq 0$.

Recursive system may be relaxed or non-relaxed, depending on the initial condition.

A system is linear if satisfy the following requirements:

1. The total response is equal to the sum of the zero-input and zero-state responses

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

2. The principles of superposition applies to the zero-state response (zero-state linear)

3. The principles of superposition applies to the zero-input response (zero-input linear)

If a system does not satisfy all three separate requirement, system is called nonlinear.

Example:

$$y(n) = ay(n-1) + x(n)$$

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k)$$

$y(n) = y_{zs}(n) + y_{zi}(n)$ satisfy requirement one.

The second requirement:

Let assume $x(n) = c_1 x_1(n) + c_2 x_2(n)$

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k)$$

$$= \sum_{k=0}^{n} a^k \left[c_1 x_1(n-k) + c_2 x_2(n-k)\right]$$

$$= c_1 \sum_{k=0}^{n} a^k x_1(n-k) + c_2 \sum_{k=0}^{n} a^k x_2(n-k)$$

$$= c_1 y_{zs}^{(1)}(n) + c_2 y_{zs}^{(2)}(n)$$

Zero=state linear.

The third requirement:

Let assume

$$y(-1) = c_1 y_1(-1) + c_2 y_2(-1)$$

$$y_{zi}(n) = a^{n+1} y(-1) = a^{n+1}\left[c_1 y_1(-1) + c_2 y_2(-1)\right]$$

$$= c_1 a^{n+1} y_1(-1) + c_2 a^{n+1} y_2(-1)$$

$$= c_1 y_{zi}^{(1)}(-1) + c_2 y_{zi}^{(2)}(-1)$$

2.4.3 Solution of Linear Constant-Coefficient Difference Equations

Two methods
Direct method
Indirect Method (z-transform)

<u>Direct solution Method:</u>

The total solution is the sum of two parts

$$y(n) = y_h(n) + y_p(n)$$

Part 1 $y_h(n)$ homogeneous solution

Part 2 $y_p(n)$ particular solution.

<u>The Homogeneous solution</u>

Assuming that the input $x(n) = 0$

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

The solution is the form of an exponential

$$y_h(n) = l^{n}$$

If we substitute this in the previous equation.

$$\sum_{k=0}^{N} a_k l^{(n-k)} = 0$$

$$l^{n} + a_1 l^{n-1} + a_2 l^{n-2} + \ldots + a_{N-1} l^{n-(N-1)} + a_N l^{n-N} = 0$$
$$l^{n-N}\left(l^{N} + a_1 l^{N-1} + a_2 l^{N-2} + \ldots + a_{N-1} l + a_N \right) = 0$$

This is called characteristic polynomial of the system. It has N roots and denotes by $l_1, l_2, ..., l_N$.

The roots can be real or complex or some roots are identical.

Let assume that roots are real and not identical, the solution becomes

$$y_h(n) = C_1 l_1^n + C_2 l_2^n + ... + C_N l_N^n$$

The coefficients $C_i, i = 1,2,...,N$ are determined from the initial conditions. Since $x(n) = 0$, this gives us the ***zero-input response*** of the system

If there are two identical roots, the solution becomes

$$y_h(n) = C_1 l_1^n + C_1 n l_1^n + C_2 l_2^n + ... + C_N l_N^n$$

Example: Find the homogenous solution and the zero-input response for

Setting $x(n) = 0$ and The homogeneous solution form $y_h(n) = l^n$

$$l^n + a_1 l^{n-1} = 0$$

$$l^{n-1}(l - a_1) = 0$$

$$l = -a_1$$

The homogenous solution of the deference equation is

$$y_h(n) = C l^n = C(-a_1)^n$$

For the zero-input response, we have

$$y(0) + a_1 y(0-1) = 0 \rightarrow y(0) = -a_1 y(-1) \text{ and}$$

$$y_h(0) = C(-a_1)^0 = C$$

Using the last two equations $C = -a_1 y(-1)$

$$y_{zi}(n) = -ay(-1)(-a_1)^n = (-a_1)^{n+1} y(-1)$$

Example: Find the zero-input response for the second-order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

The homogeneous solution form $y_h(n) = l^n$

$$l^n - 3l^{n-1} - 4l^{n-2} = 0$$
$$l^{n-2}(l^2 - 3l - 4) = 0$$

The roots are $l_1 = -1,\ l_2 = 4$.

The homogenous solution is

$$\begin{aligned} y_h(n) &= C_1 l_1^n + C_2 l_2^n \\ &= C_1(-1)^n + C_2 4^n \end{aligned}$$

For the zero-input response, we have

$$y(0) = 3y(-1) + 4y(-2)$$

$$\begin{aligned} y(1) &= 3y(0) + 4y(-1) \\ &= 3[3y(-1) + 4y(-2)] + 4y(-1) \\ &= 13y(-1) + 12y(-2) \end{aligned}$$

From the homogeneous equation

$$y(0) = C_1 + C_2$$

$$y(1) = -C_1 + 4C_2$$

So, that

$$C_1 + C_2 = 3y(-1) + 4y(-2)$$

$$-C_1 + 4C_2 = 13y(-1) + 12y(-2)$$

The coefficients are obtained as

$$C_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2)$$

$$C_2 = \frac{16}{5}y(-1) + \frac{16}{5}y(-2)$$

Then, The zero-input response is obtained

$$y_{zi}(n) = \left[-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right](-1)^n + \left[\frac{16}{5}y(-1) + \frac{16}{5}y(-2)\right]4^n$$

The Particular solution:

The particular solution is depends on the form of the input $x(n)$.

Input Signal $x(n)$	Particular Solution $y_p(n)$
A (constant)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \ldots + K_M$
$A^n n^M$	$A^n \left(K_0 n^M + K_1 n^{M-1} + \ldots + K_M \right)$
$\begin{Bmatrix} A\cos w_0 n \\ A\sin w_0 n \end{Bmatrix}$	$K_1 \cos w_0 n + K_2 \sin w_0 n$

Example:

$$y(n) + a_1 y(n-1) = x(n), \quad |a_1| < 1, \text{ the input } x(n) = u(n)$$

The particular solution is in the following form

$$y_p(n) = Ku(n)$$

where K is scale factor.

$$Ku(n) + a_1 Ku(n-1) = u(n) \rightarrow K + a_1 K = 1, \text{ forany } n \geq 1$$

Hence,

$$K = \frac{1}{1 + a_1}$$

and the particular solution is

$$y_p(n) = \frac{1}{1+a_1} u(n)$$

Example: Find the $y_p(n)$ for

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n), \quad x(n) = 2^n, n \geq 0$$

The particular solution form

$$y_p(n) = K2^n, \; n \geq 0 \quad \rightarrow y_p(n) = K2^n u(n)$$

$$K2^n u(n) = \frac{5}{6} K2^{n-1} u(n-1) - \frac{1}{6} K2^{n-2} u(n-2) + 2^n u(n)$$

Evaluate the equation for $n \geq 2$

$$4K = \frac{5}{6} 2K - \frac{1}{6} K + 4 \quad \rightarrow \quad K = \frac{8}{5}$$

Therefore, the particular solution is

$$y_p(n) = \frac{8}{5} 2^n, \; n \geq 0$$

The total solution of the difference equation

$$y(n) = y_h(n) + y_p(n)$$

Example:

$y(n) + a_1 y(n-1) = x(n)$, $n \geq 0$, the input $x(n) = u(n)$

The homogeneous solution is

$$y_h(n) = C\, l^n = C(-a_1)^n$$

The particular solution is

$$y_p(n) = \frac{1}{1+a_1}$$

The total solution is

$$y(n) = y_h(n) + y_p(n) = C(-a_1)^n + \frac{1}{1+a_1}$$

Let's find C.

For initial condition $y(-1) = 0$, gives us zero-state response of the system

$$y(0) + a_1 y(-1) = 1, \quad \rightarrow y(0) = 1$$

From the solution $y(0) = C + \frac{1}{1+a_1} = 1 \quad \rightarrow c = \frac{a_1}{1+a_1}$

Therefore,

$$y_{zs}(n) = \frac{a_1}{1+a_1}(-a_1)^n + \frac{1}{1+a_1} = \frac{1-(-a_1)^n}{1+a_1}, \quad n \geq 0$$

The condition $y(-1) \neq 0$ gives us the total solution includes zero-state and zero-input response of the system

$$y(0) + a_1 y(-1) = 1, \quad y(0) = -a_1 y(-1) + 1$$

From the solution $y(0) = C + \frac{1}{1+a_1} = 1$

$$C + \frac{1}{1+a_1} = -a_1 y(-1) + 1, \quad \rightarrow C = -a_1 y(-1) + \frac{a_1}{1+a_1}$$

$$
\begin{aligned}
y(n) &= \left(-a_1 y(-1) + \frac{a_1}{1+a_1} \right) (-a)^n + \frac{1}{1+a_1} \\
&= (-a)^{n+1} y(-1) + \frac{1-(-a_1)^{n+1}}{1+a_1} \\
&= y_{zi}(n) + y_{zs}(n)
\end{aligned}
$$