ECE 307

Fourier Series

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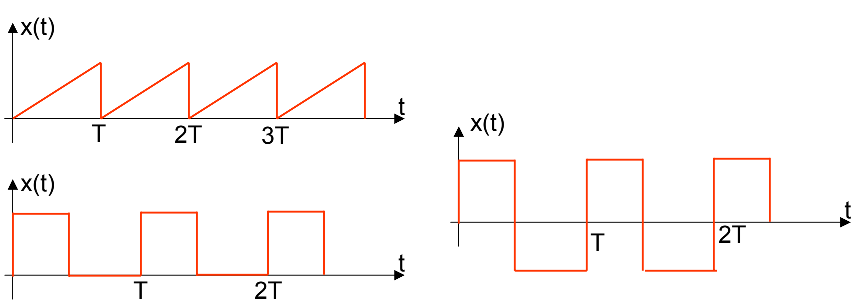
Fourier Series

Periodic signal is a function that repeats itself every T seconds.

$$x(t) = x(t \pm nT)$$

T: period of a function,   n: integer 1,2,3,…



Fourier Series

Periodic signal can be represented as sum of sinusoidals if the signal is square-integrable over an arbitrary interval ().

$$\int_{t_1}^{t_1+T} |x(t)|\,dt < \infty$$

So, it can be expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$
$$= c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$
$$= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T_0}$

ω_0 fundamental frequency of the periodic function in [rad/s].
$n\omega_0$, for $n = 2,3,4,\ldots$ are harmonic frequencies

Fourier Series

The parameters are called Fourier series expansion or coefficients and given by

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} x(t)\cos(n\omega_0 t)\,dt \qquad n = 1,2,3,\ldots$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} x(t)\sin(n\omega_0 t)\,dt \qquad n = 1,2,3,\ldots$$

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad \theta_n = -\tan\frac{b_n}{a_n} \qquad c_0 = a_0$$

$$X_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t) e^{-jn\omega_0 t}\,dt \qquad n = \mp 1, \mp 2, \mp 3, \ldots$$

Where t_1 is arbitrary. It can be set $t_1 = 0$ or $t_1 = -T_0/2$

Fourier Series

Using Euler's rule, X_n can be written as

$$X_n = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)\cos(n\omega_0 t)dt - j\frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)\sin(n\omega_0 t)dt$$

$$X_n = \frac{1}{2}a_n - j\frac{1}{2}b_n$$

If x(t) is a real-valued periodic signal, we have

$$X_{-n} = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)e^{jn\omega_0 t}dt = \frac{1}{T_0}\left[\int_{t_1}^{t_1+T_0} x(t)e^{-jn\omega_0 t}dt\right]^*$$

$$X_{-n} = X_n^*$$

$$X_{-n} = \frac{1}{2}a_n + j\frac{1}{2}b_n$$

To obtain and

$$a_n = 2\,\mathrm{Re}\{X_n\}$$

$$b_n = -2\,\mathrm{Im}\{X_n\}$$

$$X_n = \frac{1}{2}c_n e^{j\theta_n}, \quad n = 1,2,3$$

Fourier Series

Remember that

$$\int_{t_1}^{t_1+T_0} \cos(n\omega_0 t)\,dt = 0 \quad \text{for all n}$$

$$\int_{t_1}^{t_1+T_0} \sin(n\omega_0 t)\,dt = 0 \quad \text{for all n}$$

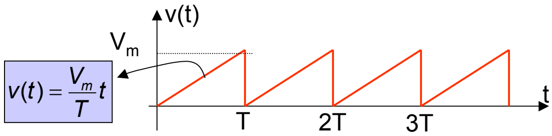
$$\int_{t_1}^{t_1+T_0} \cos(n\omega_0 t)\sin(m\omega_0 t)\,dt = 0 \quad \text{for all n and m}$$

$$\int_{t_1}^{t_1+T_0} \cos(n\omega_0 t)\cos(m\omega_0 t)\,dt = 0 \quad \text{for all } n \neq m$$
$$= \frac{T}{2} \quad \text{for all } n = m$$

$$\int_{t_1}^{t_1+T_0} \sin(n\omega_0 t)\sin(m\omega_0 t)\,dt = 0 \quad \text{for all } n \neq m$$
$$= \frac{T}{2} \quad \text{for all } n = m$$

Fourier Series

Example: Find the Fourier series of the following periodic signal



$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} x(t)dt = \frac{1}{T} \int_0^T \frac{V_m}{T} t \; dt = \frac{1}{T} \left(\frac{V_m}{2T} t^2 \right) \Bigg|_0^T = \frac{1}{2} V_m$$

$$a_n = \frac{2}{T} \int_0^T \frac{V_m}{T} t \cos(n\omega_0 t) \; dt = \frac{2V_m}{T^2} \int_0^T t \cos(n\omega_0 t) \; dt$$

$$= \frac{2V_m}{T^2} \left(\frac{1}{n^2 \omega_0^2} \cos(n\omega_0 t) + \frac{t}{n\omega_0} \sin(n\omega_0 t) \right) \Bigg|_0^T$$

$$= \frac{2V_m}{T^2} \left(\frac{1}{n^2 \omega_0^2} \cos(n \frac{2\pi}{T} T) - \frac{1}{n^2 \omega_0^2} \right) = 0 \quad \text{for all n}$$

Fourier Series

$$b_n = \frac{2}{T} \int_0^T \frac{V_m}{T} t \sin(n\omega_0 t) \; dt = \frac{2V_m}{T^2} \int_0^T t \sin(n\omega_0 t) \; dt$$

$$= \frac{2V_m}{T^2} \left(\frac{1}{n^2 \omega_0^2} \sin(n\omega_0 t) - \frac{t}{n\omega_0} \cos(n\omega_0 t) \right) \Bigg|_0^T$$

$$= \frac{2V_m}{T^2} \left(0 - \frac{T}{n\omega_0} \cos(n \frac{2\pi}{T} T) \right) = -\frac{V_m}{n\pi} \quad \text{for n}$$

The Fourier Series

$$v(t) = a_0 + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

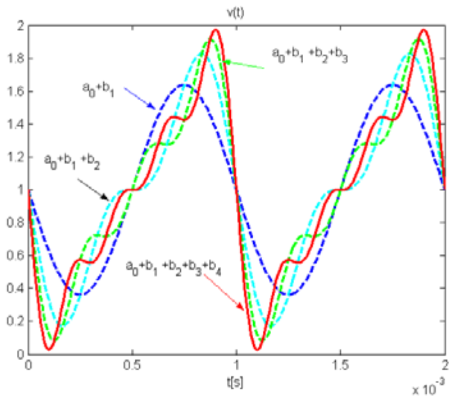
$$v(t) = \frac{V_m}{2} - \sum_{n=1}^{\infty} \frac{V_m}{n\pi} \sin(n\omega_0 t)$$

$$v(t) = \frac{V_m}{2} - \frac{V_m}{\pi} \sin(\omega_0 t) - \frac{V_m}{2\pi} \sin(2\omega_0 t) - \frac{V_m}{3\pi} \sin(3\omega_0 t) - \ldots$$

Fourier Series

Let's assume that V_m=2V and T=1ms

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi 1000 \quad \text{rad/s}$$




```
>> Vm=2;
>> T=0.001;
>> w0=2*pi/T;
>> t=0:0.00001:0.002;
>> v1=Vm/2-Vm/pi*sin(w0*t);
>> plot (t,v1)
>> hold on;
>> v2=Vm/2-Vm/pi*sin(w0*t)-
Vm/(2*pi)*sin(2*w0*t);
>> plot (t,v2)
>> v3=Vm/2-Vm/pi*sin(w0*t)-
Vm/(2*pi)*sin(2*w0*t)-
Vm/(3*pi)*sin(3*w0*t);
>> plot (t,v3)
>> v4=Vm/2-Vm/pi*sin(w0*t)-
Vm/(2*pi)*sin(2*w0*t)-
Vm/(3*pi)*sin(3*w0*t)-
Vm/(4*pi)*sin(4*w0*t);
>> plot (t,v4)
>> xlabel ('t[s]')
>> title('v(t)')
```

Fourier Series

$$b_n = -\frac{V_m}{n\pi} \quad \text{for } n=1,2,3,...$$

$$c_n = \sqrt{a_n^2 + b_n^2} = \frac{V_m}{n\pi} \quad \text{for } n=1,2,3,...$$

$$\theta_n = -\tan\frac{b_n}{a_n} = 90^\circ$$

$$v(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \theta_n)$$

$$X_n = \frac{1}{2}a_n - j\frac{1}{2}b_n = j\frac{1}{2}\frac{V_m}{n\pi}$$

$$X_n = \frac{1}{2}c_n e^{j\theta_n} = \frac{1}{2}\frac{V_m}{n\pi}e^{j90^\circ}$$

$$X_0 = c_0$$

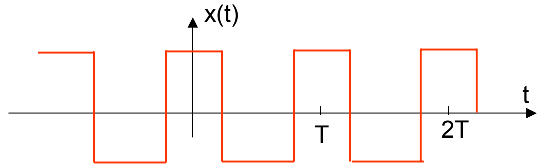
Fourier Series

The Effect of symmetry on the Fourier Coefficients

Even-function symmetry

Even-function is defined as $x(t) = x(-t)$

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)dt \qquad a_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\cos(n\omega_0 t)dt \qquad b_n = 0 \text{ for all n}$$



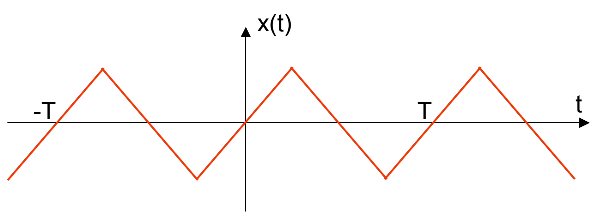
Fourier Series

The Effect of symmetry on the Fourier Coefficients

Odd-function symmetry

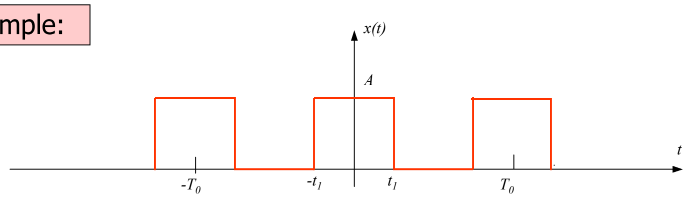
Odd-function is defined as $x(t) = -x(-t)$

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)dt \qquad b_n = \frac{4}{T_0}\int_0^{T_0/2} x(t)\sin(n\omega_0 t)dt \qquad a_n = 0 \text{ for all n}$$



Fourier Series

Example:



Assume that ,A=1 ,$T_0 = 4\,s$ and $t_1 = 1\,s$ Determine Fourier series coefficients of in exponential and trigonometric form.

Plot the discrete spectrum of x(t).

$x(t) = x(-t)$

$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)dt$$

$$a_n = \frac{4}{T_0}\int_{0}^{T_0/2} x(t)\cos(n\omega_0 t)dt$$

$b_n = 0$ for all n

Fourier Series

Example:

$$a_0 = \frac{1}{T_0}\left[\int_{-T_0/4}^{T_0/4} 1\,dt\right] = \frac{1}{T_0}\left[t\Big|_{-T_0/4}^{T_0/4}\right]$$

$$= \frac{1}{T_0}\left[\frac{T_0}{4} + \frac{T_0}{4}\right] = \frac{1}{2}$$

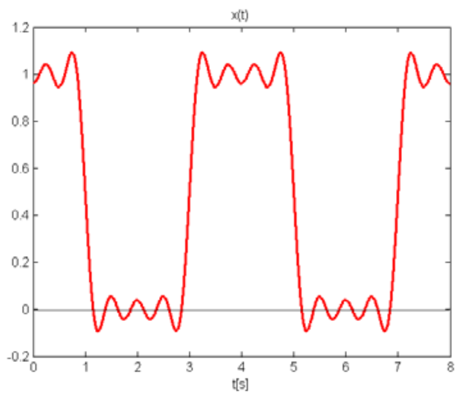
$$a_n = \frac{4}{T_0}\int_{0}^{T_0/4} 1\cos(n\omega_0 t)\,dt = \frac{4}{T_0}\left[\frac{1}{n\omega_0}\sin(n\omega_0 t)\Big|_0^{T_0/4}\right]$$

$$= \frac{4}{T_0}\left[\frac{1}{n\frac{2\pi}{T_0}}\left(\sin(n\frac{2\pi}{T_0}\frac{T_0}{4}) - 0\right)\right] = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$a_n = \left\{\frac{2}{\pi}, 0, -\frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0, -\frac{2}{7\pi}, ...\right\}$$

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t) - \frac{2}{3\pi}\cos(3\omega_0 t) + \frac{2}{5\pi}\sin(5\omega_0 t) - \frac{2}{7\pi}\cos(7\omega_0 t)...$$

Fourier Series




```
>> t=0:0.001:8;
>> T=4;
>> w0=2*pi/T;
>> v=1/2+2/pi*cos(w0*t)-2/(3*pi)*cos(3*w0*t)+2/(5*pi)*cos(5*w0*t)-2/(7*pi)*cos(7*w0*t);
>> plot (t,v)
>> xlabel ('t[s]')
>> title('v(t)')
```

Fourier Series

Example :

$$X_n = \frac{1}{4}\int_{-1}^{1} 1e^{-jn\omega_0 t}dt = \frac{1}{-j4n\omega_0}\left[e^{-jn\omega_0} - e^{jn\omega_0}\right]$$

$$= \frac{1}{2n\omega_0}\frac{1}{j2}\left[e^{jn\omega_0} - e^{-jn\omega_0}\right]$$

$$= \frac{1}{2n\omega_0}\sin(n\omega_0) = \frac{1}{2}\frac{\sin(n\omega_0)}{n\omega_0}$$

$$= \frac{1}{2}\frac{\sin(n2\pi/4)}{n2\pi/4} = \frac{1}{2}\frac{\sin(n\pi/2)}{n\pi/2}$$

$$= \frac{1}{2}\mathrm{sinc}\left(\frac{n}{2}\right)$$

where

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Fourier Series

Example.1: (cont)

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} \frac{1}{2}\text{sinc}(\frac{n}{2})e^{jn\omega_0 t}$$

Since is real and even,

$$a_n = 2|X_n| = \text{sinc}(\frac{n}{2})$$

$$a_0 = \frac{1}{2}$$

$$c_n = \left|\text{sinc}(\frac{n}{2})\right|$$

$$b_n = 2|X_n|\sin(\angle X_n) = 0$$

$$\theta_n = 0, \pi$$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \text{sinc}(\frac{n}{2})\cos(n\omega_0 t)$$

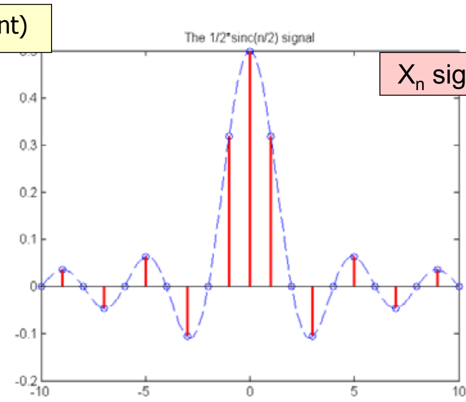
Since $\omega_0 = 2\pi / 4$

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \text{sinc}(\frac{n}{2})\cos(\frac{n\pi t}{2})$$

x(t) n=1,3, 5,…has odd numbers' harmonics. The even numbers' harmonics are zero. X_n is always real, so that the phase is either zero or π . The magnitude of discrete spectrum is shown in next page.

Fourier Series

Example.1: (cont)

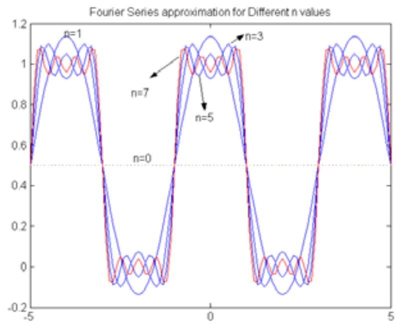


X_n signal as sinc function

```
>> n=-10:1:10;
>> x=0.5*sinc(n/2);
>> stem (n,x)
>> title('The 1/2*sinc(n/2) signal');
>> xlabel('n');
```

Fourier Series

Example.1: (cont)

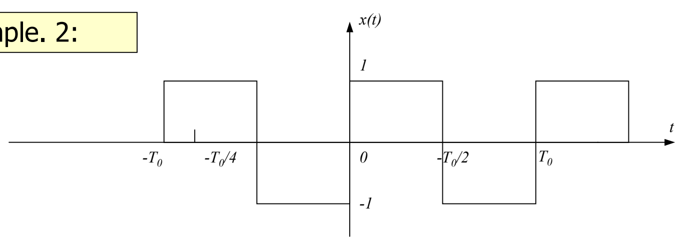


Fourier series approximation of signal x(t) for . $n = 0,1,3,5,\text{ and } 7$

```
>> t=-5:0.1:5;
>> n=0;
>> x=0.5;
>> plot (t,x)
>> hold on
>> n=1;
>> an=(sinc(n/2)*cos(2*pi*t*n/4));
>> x=x+an;
>> plot (t,x)
>> n=3;
>> an=(sinc(n/2)*cos(2*pi*t*n/4));
>> x=x+an;
>> plot (t,x)
>> n=5;
>> an=(sinc(n/2)*cos(2*pi*t*n/4));
>> plot (t,x)
>> x=x+an;
>> plot (t,x)
>> n=7;
>> an=(sinc(n/2)*cos(2*pi*t*n/4));
>> x=x+an;
>> plot (t,x,'r')
>> title('Fourier Series approximation for
Different n values')
>>
```

Fourier Series

Example. 2:



$$a_0 = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)dt = 0$$

$$X_n = \frac{1}{T_0}\int_{t_1}^{t_1+T_0} x(t)e^{-jn\omega_0 t}dt$$

$$n = \mp 1, \mp 2, \mp 3, \ldots$$

$$X_n = \frac{1}{T_0}\left[\int_{0}^{T_0/2} e^{-jn\omega_0 t}dt + \int_{T_0/20}^{T_0} e^{-jn\omega_0 t}dt\right]$$

Fourier Series

Example. 2: (cont)

$$X_n = \frac{1}{T_0}\left[\frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_0^{T_0/2} - \frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{T_0/2}^{T_0}\right]$$

$$X_n = \frac{1}{T_0}\left[\frac{1}{-jn\omega_0}(e^{-jn\omega_0 T_0/2} - e^{-jn\omega_0 0}) - \frac{1}{-jn\omega_0}(e^{-jn\omega_0 T_0} - e^{-jn\omega_0 T_0/2})\right]$$

$$X_n = \frac{1}{T_0}\frac{1}{j\frac{n2\pi}{T_0}}\left[1 - e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} + e^{-jn\frac{2\pi}{T_0}T_0} - e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}}\right]$$

$$X_n = \frac{1}{jn2\pi}\left[1 - e^{-jn\pi} + e^{-jn2\pi} - e^{-jn\pi}\right]$$

$$X_n = \frac{2}{jn2\pi}\left[1 - e^{-jn\pi}\right]$$

Fourier Series

Example. 2: (cont)

$$X_n = \frac{2}{jn2\pi}e^{-j\frac{\pi}{2}n}\left[e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right]$$

$$X_n = \frac{2}{n\pi}e^{-j\frac{\pi}{2}n}\left[\sin\frac{\pi}{2}n\right]$$

$$n = \pm 1, \pm 2, \pm 3, \ldots$$

$$X_n = e^{-j\frac{\pi}{2}n}\left[\frac{\sin\frac{\pi}{2}n}{\frac{n\pi}{2}}\right] = e^{-j\frac{\pi}{2}n}\text{sinc}\left(\frac{n}{2}\right)$$

n	X_n
1	0-j 0.6366
2	0+j0
3	0-j0. 212
4	0+j0
5	0-j0. 127
6	0+j0
7	0-j 0. 0909
8	0+j0
9	0-j0. 0707
10	0+j0
11	0-j 0. 05787

```
>> n=1:11;
>> x=(2./(pi*n)).*(sin(pi/2*n)).*exp(-j*(pi*n/2));
```

Fourier Series

Example. 2: (cont)

$$X_n = \frac{1}{2}a_n - j\frac{1}{2}b_n$$

```
bn=(-2)*imag(x)
```

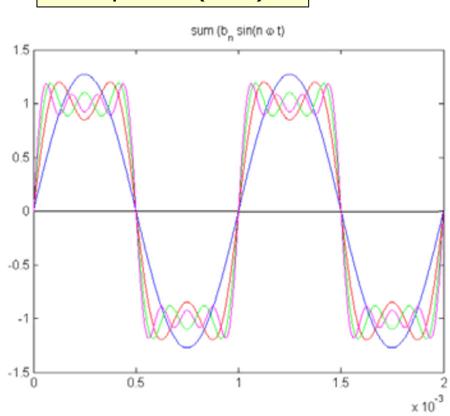
n	1	2	3	4	5	6	7	8	9	10	11
bn	1.273	0	0.4244	0	0.2546	0	0.18189	0	0.14147	0	0.1157

$$x(t) = 1.273\sin\omega_0 t + 0.4244\sin 3\omega_0 t + 0.2546\sin 5\omega_0 t + 0.1818\sin 7\omega_0 t + ...$$

$$x(t) = \frac{4}{\pi}\left[\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \frac{1}{7}\sin 7\omega_0 t + ...\right]$$

Fourier Series

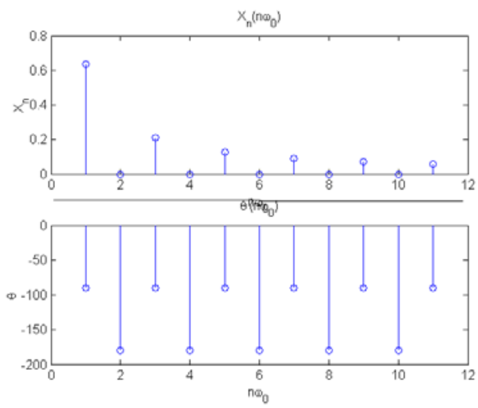
Example. 2: (cont)




```
>> t=0:0.000001:0.002;
>> b1=1.273*sin(2*pi*1000*t);
>> plot (t,b1)
>> hold on
>> b3=0.4244*sin(2*3*pi*1000*t);
>> b=b1+b3;
>> plot (t,b,'r')
>> b5=0.2546*sin(2*5*pi*1000*t);
>> b=b1+b3+b5;
>> plot (t,b,'g')
>> b7=0.18189*sin(2*7*pi*1000*t);
>> b=b1+b3+b5+b7;
>> plot (t,b,'y')
>> b9=0.14147*sin(2*9*pi*1000*t);
>> plot (t,b,'m')
>> title ('sum (b_n sin(n \omega t)')
>>
```

Fourier Series

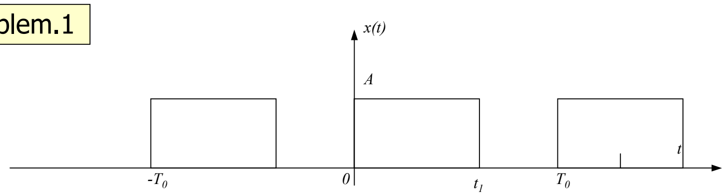
Example. 2: (cont)




```
>>n=1:11;
>>
x=(2./(n)).*(sin(pi/2*n)).*exp(j
*(pi*n/2));

xn=abs(x);
theta=(180*angle(x))/pi;

subplot (2,1,1);
stem(n,xn)
xlabel('n\omega_0');
ylabel('X_n');
title('X_n(n\omega_0)');

subplot (2,1,2)
stem(n,theta)
xlabel('n\omega_0')
ylabel('\theta')
title('\theta (n\omega_0)');
```

Fourier Series

Problem.1



Assume that ,A=1 ,$T_0 = 4\,s$ and $t_1 = 1\,s$ Determine Fourier series coefficients of in exponential and trigonometric form.

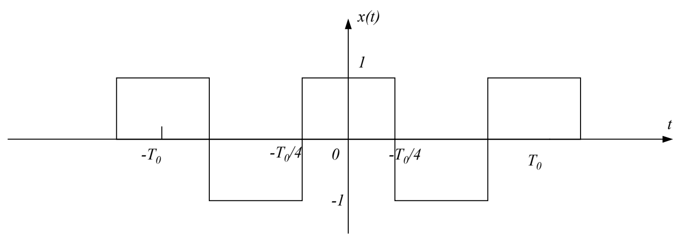
Plot the discrete spectrum of x(t).

Compare with Example 1

Fourier Series

Problem:

Write the Fourier series for the following periodic signal and plot the sum of first 10 harmonics



$$x(t) = 1.273\cos\omega_0 t - 0.4244\cos 3\omega_0 t + 0.2546\cos 5\omega_0 t - 0.1818\cos 7\omega_0 t + \ldots.$$