Notes on Voltage, Capacitance, Circuits

Voltage

At the end of the last section we calculated the work done **by** the electrostatic force when a particle of charge q is moved directly away from a fixed particle of charge Q. If r_i is the initial distance and r_f the final distance from the charge Q, then the work was found by integrating kQq/r^2 from r_i to r_f:

$$W_{electrostatic}(r_i \rightarrow r_f) = k\frac{Qq}{r_i} - k\frac{Qq}{r_f} \tag{1}$$

This result was for the special path that was a straight line away from the charge Q. We will show in class that one gets the **same result for any path** that the charge q travels between $\vec{r}_i$ and $\vec{r}_f$. this is because the electrostatic force is a central force, and hence conservative. The only contribution to the work done by the electric force is from the component of the path element that is radially away from the source charge Q. Only the initial and final distance matter in the net work done by the electrostatic force on the point charge q.

This is a very nice result, and can be generalized to any number of "point" source particles. Suppose we have N small "point" particles. Label them 1, 2, ..., i, ... N, and let them remain fixed in place. Label the charge on the i'th particle as Q_i. Since the superposition principle applies to the electric force, the work done **by** the electric force as the point charge q follows a path from $\vec{r}_a$ to $\vec{r}_b$ is given by:

$$W_{electrostatic}(r_a \rightarrow r_b) = \sum_{i=1}^{N} k\frac{Q_i q}{r_{ia}} - k\frac{Q_i q}{r_{ib}} \tag{2}$$

where r_{ia} is the initial distance from the particle to Q_i and r_{ib} is the final distance from the particle to Q_i. Notice in this equation the q can be factored out!

$$W_{electrostatic}(r_a \rightarrow r_b) = q\sum_{i=1}^{N}(k\frac{Q_i}{r_{ia}} - k\frac{Q_i}{r_{ib}}) \tag{3}$$

This is possible since the electric force is proportional to the charge on an object. From this equation we see that it is useful to consider the work done by the electrostatic force **per charge**. We call this quantity the **electric potential difference** between the points $\vec{r}_a$ and $\vec{r}_b$. It is also refered to as the **voltage difference**, $V(\vec{r}_a) - V(\vec{r}_b)$, between the two points.

$$W_{electrostatic}(r_a \rightarrow r_b) = q(V(\vec{r}_a) - V(\vec{r}_b)) \tag{4}$$

Comparing with the equation above, we have

$$V(\vec{r}_a) = \sum_{i=1}^{N} k \frac{Q_i}{r_{ia}} \tag{5}$$

where the formula for $V(\vec{r}_b)$ is the same with a replaced by b. $V(\vec{r}_a)$ is called the electric potential at the position $\vec{r}_a$. Usually the subscript a is omitted, and the expression for the electric voltage at the position $\vec{r}$ is

$$V(\vec{r}) = \sum_{i=1}^{N} k \frac{Q_i}{r_i} \tag{6}$$

where r_i is the distance between the position $\vec{r}$ and the point charge Q_i. If there is only one source charge, then $V(\vec{r}) = kQ/r$ where r is the distance from the position $\vec{r}$ to the charge Q. If there is a continuous distribution of charge, then the sum becomes an integral:

$$V(\vec{r}) = \int k \frac{\rho(\vec{r}')dV'}{|\vec{r} - \vec{r}'|} \tag{7}$$

In class we will do examples for which we calculate the voltage and voltage difference for various point and continuous charge distributions.

Comments

1. An important application of voltage difference is that it is directly related to the work done by the electric force on the charge q. If $\Delta V \equiv V(\vec{r}_a) - V(\vec{r}_b)$ then $q\Delta V$ is the work done on the charge q if it travels from point a to point b. The energy the particle acquires is simply $q\Delta V$. Often this energy gain is equal to the particles gain in K.E.

2. $V(\vec{r})$ is defined at a point in space, the point at the position $\vec{r}$. Nothing needs to be at the point. This concept is similiar to the electric field. The electric field is the force per charge; **The voltage is the potential energy per charge**.

3. Although $\vec{r}$ is a vector, a position vector, the voltage is a **scalar**. Since voltage is a scalar, it is much easier to calculate than the electric field. There is no direction involved in using the superposition principle.

4. Only voltage differences are measurable. Our expression for the voltage due to a point particle of magnitude Q, kQ/r, takes as a reference $V(\infty) = 0$.

One can always calculate the voltage at a point in space by summing up the expression above over all the "point" particles in the system. However, it is often easier to express the voltage difference in terms of the electric field. This can be done as follows. First start with the expression for the work done by the electrostatic force for a path going from $\vec{r}_a$ to $\vec{r}_b$:

$$W_{electrostatic}(\vec{r}_a \to \vec{r}_b) = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \cdot d\vec{r} \tag{8}$$

where the integral is over the path from $\vec{r}_a$ to $\vec{r}_b$. Since the electrostatic force is conservative, one obtains the same result for any path between the two points. Since $\vec{F} = q\vec{E}$, we have

$$W_{electrostatic}(\vec{r}_a \to \vec{r}_b) = q \int \vec{E} \cdot d\vec{r} \tag{9}$$

However, the left side of the equation is just the difference in voltage ΔV times q. Thus, the q's cancel and we are left with

$$\Delta V = \int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{r} \tag{10}$$

or

$$V(\vec{r}_a) - V(\vec{r}_b) = \int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{r} \tag{11}$$

This equation is also written as

$$V(\vec{r}_b) - V(\vec{r}_a) = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{r} \tag{12}$$

We will do a number of examples in which we calculate the difference in electric potential by integrating $\vec{E} \cdot d\vec{r}$ over a path. The right side of the equation is called a line (or path) integral.

If the electric field is known at every point in space, one can use the above equation to calculate the voltage difference between any two points. Thus, from knowledge of the electric field, the electric potential can be determined. What about the converse? If the voltage is known at every point in space, can one determine the electric field? The answer is yes. Since V is obtained by integrating $\vec{E}$, then you might guess that

$\vec{E}$ is related to the derivative of V. However, $\vec{E}$ is a vector, and V is a scalar. We will probably need three derivatives of V, one for each component, to find all three components of $\vec{E}$. Consider first E_x.

Let's choose a path for the above equation to be on the x-axis. If we choose b to be at $x + \Delta x$ and choose a to be at x, then the above equation for the voltage difference between these two positions is:

$$V(x + \Delta x) - V(x) = -E_x \Delta x \tag{13}$$

Dividing by Δx and taking the limit as $\Delta x \rightarrow 0$ we have:

$$E_x = lim_{\Delta x \rightarrow 0} - \frac{V(x + \Delta x) - V(x)}{\Delta x} \tag{14}$$

which yields

$$E_x = -\frac{\partial V}{\partial x} \tag{15}$$

The partial derivative is used, since V depends on x, y, and z. When one takes the partial derivative with respect to x, y and z are held constant. The same procedure can be done with the y- and z- directions:

$$\begin{aligned} E_x &= -\frac{\partial V(x, y, z)}{\partial x} \\ E_y &= -\frac{\partial V(x, y, z)}{\partial y} \\ E_z &= -\frac{\partial V(x, y, z)}{\partial z} \end{aligned}$$

In summary, we have discussed two ways to find the electric potential at a point, or electrical potential difference between two points:

1) use the formula for a point charge source kQ/r and sum up (or integrate) over the source charges

2) First determine $\vec{E}$ and carry out the line integral $\int \vec{E} \cdot d\vec{r}$. We will do examples in class, and I present an example of each method here in the notes.

Electric Potential on the axis of a Charged Ring

Suppose we have a uniformly charged ring of radius R and total charge Q. Lets find the electric potential V on the axis of the ring a distance x from the center. For this problem, it is best to "chop" up the ring into small pieces, and sum up the contribution from each piece. Let the pieces be labeled as 1, 2, 3, ... , i, ..., and ΔQ_i be the charge on the i'th piece. The potential at the position x due to the i'th piece, ΔV_i, is just $k\Delta Q_i/\sqrt{x^2+R^2}$. This is true since each small piece can be treated as a point particle and the distance from x to the piece is just $\sqrt{x^2+R^2}$ via Pythagorus Theorum.

$$\Delta V_i = k\frac{\Delta Q_i}{\sqrt{x^2+R^2}} \tag{16}$$

Note: **there is no direction to** ΔV_i, potential is a scalar. When we add up the contributions of the pieces over the whole ring, x and R do not change, thus

$$V = \frac{k}{\sqrt{x^2+R^2}}\sum \Delta Q_i \tag{17}$$

The sum over the ΔQ_i just adds up to Q:

$$V(x,0,0) = k\frac{Q}{\sqrt{x^2+R^2}} \tag{18}$$

We can differentiate the above expression to find the electric field on the axis:

$$\begin{aligned} E_x &= -\frac{\partial V}{\partial x} \\ &= \frac{kQx}{(x^2+R^2)^{3/2}} \end{aligned}$$

Note: this is the same result we obtained by using Coulomb's Law for the electric field and integrating over the ring. Find the electric potential first and differentiating was easier, and often is. Sometimes it is easier to find $\vec{E}$ first and then V, and sometimes it is easier to determine V first and differentiate to obtain $\vec{E}$.

Potential Difference between two parallel plates

Suppose we have two flat identical plates, each with an area A. Suppose the plates are parallel to each other and separated by a distance d. Lets also assume that

$\sqrt{A} >> d$, that is the plates are very large compared to the distance separating them. Lets place a total of $+Q$ amount of charge on one plate and a total of $-Q$ on the other. We spread the charge uniformly over the surface area A. What is the electric potential difference (voltage difference) from one plate to the other?

In this case it is easiest to first determine the electric field between the plates and carry out $\int \vec{E} \cdot d\vec{r}$. We already determined the electric field from an infinite plate using Gauss' Law as well as using Coulomb's Law and integrating over the plate. Since $\sqrt{A} >>> d$, we can approximate the plates as infinite. Previously we found the electric field between the parallel plates by adding up the electric field from each plate: $|\vec{E}| = Q/(2A\epsilon_0) + Q/(2A\epsilon_0)$. So the net electric field between the plates is $|\vec{E}| = Q/(A\epsilon_0)$ and is **constant** between the plates. Since the electric force is conservative, we can take calculate the potential difference using any path from one plate to the other. Lets choose a path straight across, and since $\vec{E}$ is constant, the line integral is very easy.

$$
\begin{aligned}
V &= \int \vec{E} \cdot d\vec{r} \\
&= Ed \\
V &= \frac{Qd}{A\epsilon_0}
\end{aligned}
$$

The line integral is simple for the following reasons: 1) since $\vec{E}$ and $d\vec{r}$ are in the same direction, the dot product is just the produce of the magnitude of $\vec{E}$ and $d\vec{r}$. 2) Since $\vec{E}$ is constant, the line integral is just $|\vec{E}|$ times the distance between the plates.

Remember that the voltage difference gives us the electrical potential energy change **per charge** of a particle. That is, if a particle of charge q travels through a potential difference V, the work done by the electrical force is $W = qV$. **If the electrical force is the only force that does work on the particle**, then the gain in K.E. is qV so

$$qV = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} \tag{19}$$

If the particle starts from rest, it attains a speed $v = \sqrt{2qV/m}$. If an electron travels through a voltage difference of one volt, the work done on it by the electrical force is 1.6×10^{-19}C (1)Volt $= 1.6 \times 10^{-19}$ Joules. This unit of energy is called an electron-volt or eV. 1eV $= 1.6 \times 10^{-19}$ Joules.

Capacitance

Capacitance is a property of **two conductors**. Suppose we have two conductors (metals) that are initially uncharged. Suppose we then put a total charge of $+Q$ on one and a total charge of $-Q$ on the other. This can be accomplished by transfering electrons from one to the other. If we wait for the charges (electrons) to come to static equilibrium, the electric fields within each conductor will be zero. This means that the whole conductor is at the same electrical potential. At points outside the conductors there will be electric fields, and $\int \vec{E} \cdot d\vec{r}$ from one conductor to the other will be the potential difference V between the conductors.

Now, there is a nice property about the potential difference. If we were to transfer twice as much charge, the electric fields everywhere would double and so would the potential difference V. If three times as much charge is transfered, the electric fields would triple and so would V. That is, V is proportional to $\vec{E}$ since $V = \int \vec{E} \cdot d\vec{r}$, and $\vec{E}$ is proportional to Q from Coulomb's law. Thus the voltage between two conductors with equal and opposite charge is proportional to the magnitude of the charge on them:

$$V \propto Q \tag{20}$$

This can also be written as

$$Q \propto V \tag{21}$$

Replacing the proportional sign with an equal sign and a constant we have:

$$Q = CV \tag{22}$$

where the constant C is called the capacitance of the two conductors. A unit of capacitance is Coulomb/Volt. The capacitance depends only on the geometry of the two conductors.

Lets calculate the capacitance of two parallel plate conductors. Let each plate have an area A and let the plates be separated by a distance d.

Recipe for calculating capacitance:

a) first place a charge of $+Q$ on one plate and a charge of $-Q$ on the other
b) determine $\vec{E}$ for points between the conductors
c) calculate $V = \int \vec{E} \cdot d\vec{r}$
d) divide V by Q to find C.

Parallel Plate Capacitor

For the parallel plate conductors, we already did this and found $V = Qd/(A\epsilon_0)$. Thus, the capacitance for the parallel plates is $C = A\epsilon_0/d$. Note that C only depends on the geometry of the conductors and not on any charge or voltage they might have. **If** the conductors have a potential difference of V, **then** they each acquire a charge of CV.

Coaxial Cylinderical Capacitor

Consider two cylindrical shells that are coaxial. Each shell is a conductor. Let them both have a length l, and let one have a radius a and the other a radius $b > a$. Lets also assume that $l >> (b - a)$, so we can treat the capacitor as being infinitely long. Now, let's find the capacitance of these two conductors.

First, suppose there is a total charge of $+Q$ on the inner most conductor, and a charge of $-Q$ on the outer one. Next, we need to find the electric field in the space between the conductors. Since $l >> (b - a)$ we can make the approxmation that the conductors are infinitely long. In that case the electric field between the conductors must point away from the axis of the conductors. We used Gauss' Law previously for the case of an infinitely long rod, and the same analysis will apply here. We can choose as a mathematical surface a "can" that is co-axial with the cylinders of length d and radius r: $a < r < b$. The flux through the ends is zero. Applying Gauss' Law for the "can" surface gives:

$$2\pi r d\ E = \frac{Q(d/l)}{\epsilon_0} \tag{23}$$

Canceling the length of the can d, we obtain the magnitude of the electric field between the conductors to be:

$$E = \frac{Q}{2\pi\epsilon_0 l r} \tag{24}$$

where the direction is radial. Next we need to calculate the voltage difference between the conductors: $\Delta V = \int \vec{E} \cdot d\vec{r}$. If we take as a path for the integral a path radially away from the axis then $\vec{E}$ and $d\vec{r}$ are parallel, the line integral becomes:

$$\int \vec{E} \cdot d\vec{r} = \int_a^b \frac{Q}{2\pi\epsilon_0 l r} dr \tag{25}$$

Carrying out the integral gives:

$$V = \frac{Q}{2\pi\epsilon_0 l} ln(b/a) \tag{26}$$

where V is the potential difference between the conductors. Notice that V and Q are proportional to each other. If V is doubled, so must the charge on the conductors also double. The capacitance is the ratio of Q/V, which gives:

$$C = \frac{2\pi\epsilon_0 l}{ln(b/a)} \tag{27}$$

Comments

1. In both cases the capacitance of the pair of conductors only depends on the geometry of the conductors. Not on how much charge or voltage they might have.

2. In both cases the capacitance equals a distance times ϵ_0. In metric units $\epsilon_0 = 8.85 \times 10^{-12}$ $\mathrm{C}^2/(\mathrm{N\ m}^2)$. These units are equal to F/m (Farads per meter). Since ϵ_0 is so small, it is difficult to build a capacitor with a large value for C in Farads using small materials. One needs to use a dielectric material between the conductors.

Capacitors in Circuits

Capacitors are important in electrical applications, and we now want to consider how they behave in simple circuits. We will first consider their properties when they are placed in parallel and in series.

Circuit elements are connected to each other by wires which are themselves conductors. There is very little energy loss through the conducting wires, so a good rule for **steady state circuits** is the following: *Any two points in a circuit that are connected by wires only are at the same electrical potential.* By steady state circuit we mean one in which the currents are not changing in time and there are no changing magnetic fields through the circuit. In the next section we will consider the effects of changing magnetic fields. However, we will first start with circuits for which these effects are small and can be ignored.

Capacitors connected in Parallel

Consider two capacitors whose right sides are connected together by a wire and whose left sides are connected together by another wire. Call one "1" and the other "2". Let the capacitance of "1" be C_1, and the capacitance of "2" be C_2. Since the left sides are connected together by a conducting wire they are at the same electrical potential. Since the right sides are connected together by a wire they are also at the same electrical potential. Thus, $V_1 = V_2 \equiv V$. If the electrical voltage across each of the capacitors is V, then there will be a charge of $Q_1 = C_1 V$ on the plates of capacitor "1". There will be a charge of $Q_2 = C_2 V$ on the plates of capacitor "2". The net charge on both plates is just the sum $Q_{net} = Q_1 + Q_2$ which is $Q_{net} = V(C_1 + C_2)$. Since the capacitance is Q_{net}/V, we see that the effective (or equivalant) capacitance of two capacitors in parallel is $C_1 + C_2$:

$$C_{equiv} = C_1 + C_2 \tag{28}$$

for capacitors in parallel.

Capacitors connected in Series

Consider two parallel plate capacitors called "1" and "2". Let the right side of "1" be connected to the left side of "2". The left side of "1" and the right side of "2" are open. Suppose the capacitors are initially uncharged. Now transfer an amount of charge $+Q$ from the right side of "2" to the left side of "1". The left side of "1" will have a charge of $+Q$ and the right side of "2" will have a charge of $-Q$. What about the other sides of the capacitors?

Since the left side of "2" is a conductor, the electric field inside must be zero. For this to happen, a charge of $+Q$ must come to the left side of "2". This will leave a charge of $-Q$ on the right side of "1". Thus, when equilibrium is reached, there will be a charge of $+Q$ on the left and a charge of $-Q$ on the right sides *of each capacitor*. If two capacitors in series are initially uncharged, when charged **the same amount of charge will be on the plates of each capacitor**.

Let V_1 be the voltage difference between the plates of "1" and V_2 the voltage difference between the plates of "2". If we evaluate $\int \vec{E} \cdot d\vec{r}$ from the left side of "1" to the right side of "2" we will obtain $V_1 + V_2$. That is, the potential difference across both capacitors, V, will equal the sum of the potential differences across each capacitor.

$$V = V_1 + V_2 \tag{29}$$

where V_1 is the potential difference across "1" and V_2 the potential difference across "2". However, $V_i = Q/C_i$, so

$$V = \frac{Q}{C_1} + \frac{Q}{C_2} \tag{30}$$

The equivalent (or effective) capacitance of the two capacitors in series is $C_{equiv} = Q/V$ or

$$C_{equiv} = \frac{1}{(1/C_1) + (1/C_2)} \tag{31}$$

This equation is usually written as:

$$\frac{1}{C_{equiv}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{32}$$

The main ideas in deriving these equations for combining capacitors is that 1) for capacitors in parallel the voltage is the same across both capacitors; and 2) for capacitors in series the charge on each capacitor is the same (assuming they started out uncharged).

Current

Consider a parallel plate capacitor with capacitance C. Suppose we put a charge Q on one plate and a charge $-Q$ on the other plate. The charges will stay on the plates for a long time (ideally forever). If we now connect a wire (conductor) from one plate to the other, what will happen? Now we have one big conductor, and charge will flow until both sides are neutral. Actually electrons will flow from the negative side to the positive. We will do a demonstration in lecture that will demonstrate the charge (electrons) flowing. The flow of charge is called **electric current**.

In many applications the electrons flow through wires. If the charge (or electrons) flow in a "thin" wire, we can count the number of coulombs that pass a point in the wire per unit time, and can describe the current in terms of Coulombs/sec. A Coulomb/sec is defined as an **Ampere** or simply an Amp. The current in a wire is often given the symbol I.

If the charge is not flowing in a wire, then one needs a more general expression. The expression needs to specify the direction of the charge flow as well as the amount of charge flowing. Borrowing the concept from fluid flow, a useful definition would be the charge density, ρ times the velocity vector. A good name for this quantity is **current density** and is given the symbol $\vec{J}$:

$$\vec{J} \equiv \rho \vec{v}_d \tag{33}$$

where ρ is the charge density and $\vec{v}_d$ is the **drift velocity vector** of the charge. Note that the current density is defined **at a point**. A wire is not needed. However, if the current is flowing in a wire, then $\vec{J}$ will point in the direction of the wire. The current I in a wire is related to $|\vec{J}|$ in the wire. The amount of charge ΔQ that flows past a point in a time Δt is just $\rho v_d \Delta t\ A$, where v_d is the drift speed of the charge and A is the area of the wire:

$$\Delta Q = \rho v_d A \Delta t \tag{34}$$

Dividing by Δt and A:

$$\frac{\Delta Q/\Delta t}{A} = \rho v_d \tag{35}$$

Taking the limit as $\Delta t \to 0$, we have

$$\frac{I}{A} = \rho v_d = |\vec{J}| \tag{36}$$

Thus for a wire, the magnitude of the current density is the current/area.

Another way of expressing the current density vector is to use the number of "charge carriers" per volume, n_q. In terms of n_q the charge density is $\rho = n_q q$ where q is the charge on each "charge carrier". With this notation,

$$\vec{J} = n_q q \vec{v}_d \tag{37}$$

If electrons are the charge carriers, then n_q is the number of electrons per volume, n_e, and the charge on each electron is e: $\vec{J} = n_e e \vec{v}_d$.

Resistance

When we connected the plates of the capacitor with a wire, the charge flowed from one side to the other and the capacitor became neutral again. We "discharged" the capacitor. If we discharge the capacitor with a more "resistive" wire it takes longer for the capacitor to discharge. The current is less, there is more **electrical resistance** to the flow of charge.

How can we quantify electrical resistance? Resistance is a property of the medium (or material) that the charge is flowing in. For a material to have low resistance, charges must move more easy than a material with high resistance. What makes the charges move? An electrical force, i.e. an electric field. So one can quantify the resistivity of a material by how fast the charges "drift" for a particular electric field. For a given electric field $\vec{E}$ a material with a large resistivity will have a small current density. We can define the resistivity of a material as the ratio of $|\vec{E}|/|\vec{J}|$. The symbol for resistivity is usually ρ, however this is often used to denote density, so we will use ρ_e for electrical resistivity. This is often written as:

$$\vec{J} = \frac{1}{\rho_e}\vec{E} \tag{38}$$

Another useful quantity is the conductivity of a material. The lower the resistance, the larger the conductivity: conductivity = 1/resistance. Conductivity is usually given the symbol σ, but since we also have been using σ as surface charge density, we will call conductivity σ_e:

$$\sigma_e = \frac{1}{\rho_e} \tag{39}$$

With this definition,

$$\vec{J} = \sigma_e\vec{E} \tag{40}$$

Both ρ_e and σ_e are intrinsic properties of a material. In the table below we list values of the resistivity for some common conductors at room temperature:

element:	Cu	Al	Fe	Ag	Au
$\rho_e \times 10^{-8}$ (Ω/m)	1.7	2.82	10	1.59	2.44

The above microscopic property can be used to determine the electrical properties of a large (macroscopic) piece of the material. Suppose the material is a long cylinder, like a wire. Let the cross sectional area be A and the length be l. If the voltage difference between the ends of the material is V, then the average electric field inside is $E = V/l$. If the electric field is V/l, then the current density is $J = \sigma_e V/l$. Since the current I in the conductor (wire) is JA, we have

$$I = \frac{\sigma_e A}{l}V \tag{41}$$

This equation is usually written with V on the left side of the equation and in terms of ρ_e instead of σ_e:

$$V = \frac{\rho_e l}{A}I \tag{42}$$

The quantity $\rho_e l/A$ is called the resistance of the object and given the symbol R:

$$V = RI \tag{43}$$

The resistance of the object is proportional to its length and inversely proportional to its area: $R = \rho_e l/A$.

The above equation allows one to measure the resistance of an object: establish a voltage difference V across the object and measure the current I that flows. The resistance is $R = V/I$. The question we can ask is: Does R depend on the amount of current I that is flowing through the material? In general, the answer to this question is YES. Often, when more current flows through a material, the material gets hot and the resistance increases. However, for some materials the effects of temperature are small enough that the resistance is approximately constant over a wide range of currents. These kinds of materials are called **Ohmic Materials**.

For Ohmic materials, $V = RI$ where R is constant over a wide range of currents. In this course, we will consider currents through Ohmic materials, for which R is constant.

Capacitor Discharge

If we connect the plates of a charged capacitor with a wire of fixed resistance R, we can determine how the charge on the capacitor changes in time. Suppose we have a capacitor of capacitance C, with an initial charge Q_0. At $t = 0$ we connect the plates with a wire of resistance R. What is the charge on the capacitor as a function of time $Q(t)$? We can't write an algebraic equation for $Q(t)$, but we can write an equation that describes how $Q(t)$ changes in time. This can be done as follows:
The voltage across the capacitor and resistor, V, is related to the current I through the resistor as:

$$V = RI \tag{44}$$

We now express V and I in terms of the charge Q on the capacitor: $V = Q/C$ and the current I is the rate at which the charge is leaving the capacitor: $I = -dQ/dt$. The minus sign is because if Q is decreasing the current is positive. Substituting into the equation above and re-arrainging terms gives

$$R\frac{dQ}{dt} = -\frac{Q}{C} \tag{45}$$

This is a differential equation for Q. It is an equation relating the rate at which Q changes to how much charge Q is on the plates. The more charge, the faster it leaves the plates. It is often the case in physics that equations pertaining to the changes in quantities take on a simple form. This is why calculus is so important in physics. Capacitor discharge is such a case.

We will discuss the solution in class. We will show that $Q(t)$ is of exponential form:

$$Q(t) = Q_0 e^{-t/(RC)} \tag{46}$$

The quantity RC is called the time constant and is the time it takes for the charge to decrease to $1/e$ of its initial value. Since the voltage across the capacitor is proportional to Q, $V = Q/C$, the voltage across the capacitor also decreases exponentially in time:

$$V(t) = V_0 e^{-t/(RC)} \tag{47}$$

where V_0 is the voltage at time $t = 0$. Experiments on discharging capacitors verify the exponential decay, indicating that the resistance used does not depend on the current through it.

Circuits in General

Capacitors are nice for storing charge as well as other applications, but after the charge has left the plates no more current can flow. If one wants to sustain the current then a battery or other form of power supply must be used. In the case of a battery, a chemical reaction keeps charge available for the circuit. The potential across the battery is determined by the chemical reaction, and therefore as charge flows from one side of the battery to the other side the potential remains constant. **An ideal battery keeps a constant electrical potential difference across its terminals**, while "supplying" as much charge as necessary to flow in the circuit. The value of the potential difference is determined by the chemical elements used in the battery.

There are only two basic principles needed to analyze **steady state circuits**. These are called Kirchoff's Law's:

1. **The sum of the currents going into a junction (of wires) equals the sum of the currents leaving that junction**. Another way is to say that the charge flowing into equals the charge flowing out of any junction. This is essentially a statement that charge is conserved.

2. **The sum of the electrical potential differences around any closed path is zero**. Another way is to say this is that the sum of the voltages around a closed path is zero. In terms of the electric field: $\oint \vec{E} \cdot d\vec{r} = 0$. This is essentially a statement that the electrostatic force is a conservative force. We need to emphasize that this is only

true for circuits that don't have changing magnetic fields. We will discuss the corrections to this "law" for changing currents and/or magnetic fields in the next section.

With these two "laws" we can determine the current through any circuit element as well as the voltage differences between any two points in a circuit once we know the properties of the circuit elements. We will do examples in lecture with circuits made up of just batteries and resistors, as well as some that contain capacitors.

In some cases, we can find the effective resistance for certain combinations of resistors.

Resistors connected in Parallel

Resistors in parallel have their left sides connected together and their right sides connected together. The current that comes into the parallel set-up will split up and divide through each resistor. Suppose we have two resistors in parallel with resistance R_1 and R_2. Label the current through R_1 as I_1 and the current through R_2 as I_2. If the current that comes into the combination is I, then $I = I_1 + I_2$.

Since the right sides are connected together as well as the left sides, the potential difference V is the same across each resistor. Thus, $I_1 = V/R_1$ and $I_2 = V/R_2$. If the voltage across the resistors is V and the total current I, then the effective resistance of the two resistors is just $R_{equiv} = V/I$, or $I = V/R_{equiv}$. Starting with

$$I = I_1 + I_2 \tag{48}$$

and substituting for the current the voltages divided by the resistances:

$$\frac{V}{R_{equiv}} = \frac{V}{R_1} + \frac{V}{R_2} \tag{49}$$

dividing out the V's:

$$\frac{1}{R_{equiv}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{50}$$

If there are more than 2 resistors in parallel the method can be extended:

$$\frac{1}{R_{equiv}} = \frac{1}{R_1} + \frac{1}{R_2} \cdots \tag{51}$$

For resistors in parallel, each resistor has the potential difference across it. Resistors in parallel reduces the resistance. The equivalent resistance for resistors connected in parallel is less than the resistance of any one resistor.

Resisters connected in Series

Resistors connected in series are connected like train cars, one after the other. In this case, the current through each resister is the same: I. The voltage difference across the series of resistors is the sum of the voltage differences across each resistor. If V_i is the voltage across the i'th resistor, then

$$V = V_1 + V_2 + V_3 \cdots \tag{52}$$

where V is the voltage across the series of resistors. If the equivalent resistance of the series is R_{equiv}, then

$$R_{equiv} I = R_1 I + R_2 I + R_3 I \cdots \tag{53}$$

Since the current I is the same through each resistor, we have

$$R_{equiv} = R_1 + R_2 + R_3 \cdots \tag{54}$$

For resistors connected in series the current is the same through each one. Placing resistors in series increases the resistance in the circuit.

Energy considerations in Electric Circuits

One of the main ways that energy is transfered to our homes is via electrical circuits, thus the study of energy transfered in electrical circuits is of upmost importance. We consider first the energy transfered by a battery, then we consider resistive elements.

Energy transfer from a Battery

Consider the flow of charge from one side to the other in a battery. The battery supplys a certain amount of charge at a specific potential difference V. Suppose after a time interval Δt an amount of charge ΔQ flows from one side of the battery to the other side. When an amount of charge ΔQ moves through a potential difference of V, the energy it acquires is ΔQV. Thus, in a time interval Δt an amount of energy ΔQV is supplied by the battery. The energy per unit time is therefore, $\Delta QV/\Delta t$. The energy per unit time is the power. So the power P of the battery is

$$P = \frac{\Delta QV}{\Delta t} \tag{55}$$

$\Delta Q/\Delta t$ is the current flowing out of and into the batter, so

$$P = IV \tag{56}$$

Current times voltage is the rate at which a battery is transfering energy to the circuit. The energy comes from the stored chemical energy in the battery.

The capacity of a battery is often given in units of Amp-hours. For example, D cell batteries have a capacity of around 10 amp-hours. This means that the battery can supply a current of 1 amp for 10 hours, or 1/2 amp for 20 hours, etc. Since an amp is a Coulomb/sec, 10 amp-hours is 10 Coulombs/sec times 3600 sec = 36000 Coulombs. The voltage across a D cell is 1.5 Volts, so the initial energy stored in a typical D cell is $(36000)(1.5) = 54000$ Joules. In general, if AH is the capacity of a battery in units of amp-hours, then the stored energy is just $(AH) \times 3600 \times V$ where V is the voltage of the battery. The table below gives approximate values for some common batteries.

Type	Internal Resistance (Ω)	Voltage (V)	Capacity(Amp-hours)
AAA	0.6	1.5	0.6
AA	0.4	1.5	1.4
C	0.2	1.5	4.5
D	0.1	1.5	10
Alkaline	2	9	0.5

Energy transfer in a Resistor

Consider a resistor of resistance R with a steady current I flowing through it. As with the battery, the energy transfered to an amount of charge ΔQ flowing through the resistor is equal to the potential difference V times ΔQ. If the charge is flowing at a steady rate, then the speed with which it exits equals the speed at which it entered. Thus, the charge does not gain any kinetic energy. Where does the energy go? If the charge were subject to an electric field without any material, then it would accelerate and speed up. However, the material slows it down and keeps the charge flowing at a constant speed. Thus, the energy given to the flowing charge is transfered to the material of the resistor. The resistor gains internal energy and its temperature increases.

As with the battery, in a time Δt an amount of energy equal to ΔQV is transfered to the resistor. Therefore, the energy per unit time transfered to the resistor is $\Delta QV/\Delta t$. Since $\Delta Q/\Delta t$ is the current flowing through the resistor, we have

$$Energy\ Transfer\ rate = VI \tag{57}$$

The energy transfered to the resistor makes the resistor's temperature increase until it transfers energy to the environment in terms of internal energy transfer (heat), light, or other forms of energy. The above equation can be written in terms of the resistance and current:

$$Energy\ Transfer\ rate = I^2R \tag{58}$$

or in terms of R and V:

$$Energy\ Transfer\ rate = \frac{V^2}{R} \tag{59}$$

Energy loss in wires is an important consideration in power lines. To minimize the energy loss, or I^2R loss, the current should be a minimum. Since power is VI, this means that to minimize the energy transfered to the environment (i.e. loss) one should use a LARGE voltage and a small current in the transmission lines. This way $V \times I$ can be large, but I^2R small.

We close this section with some interesting applications of electrostatics and circuits.

Cost of Electrical Energy

Each month we get a bill for the energy we have used in our homes that was transfered through wires to our houses. The electric company monitors the amount of energy we use. The unit that they use is the kilowatt-hour. This might not seem like an energy unit, but it is. A kilowatt is a rate of energy use: $1\ kW = 1000\ Joules/sec$. An hour (3600 sec) is a measure of time, and a rate times a time is an amount:

$$\begin{aligned} 1\ kW - hr &= 1000\frac{Joules}{sec} \times 3600\ sec \\ &= 3,600,000\ Joules \end{aligned}$$

This seems like a lot of energy, but a Joule is a rather small amount of energy on a human scale. Your body uses about 100 Joules/sec to keep itself going. So a Kw-hr is a fair amount of energy. The cost for a Kw-hr of energy varies from place to place, but is around 14 cents per Kw-hr. That is a pretty small price for $3,600,000$ Joules. Think how much you pay for the $54,000$ Joules that are contained in a D-cell battery.

To calculate how much it costs to use an appliance, just multiply the Kw-hr's times the number of hours and then multiply by 14 cents.

Energy Stored in a Capacitor

When a capacitor is charged with an amount of charge $+Q$ on one plate and $-Q$ on the other plate it has a certain amount of electrical potential energy. When the capacitor discharges, say through a light bulb, one sees the transfer of this energy. The question we want to consider is how much potential energy does a capacitor of capacitance C have when it is charged?

You might guess that the potential energy stored in a charged capacitor is just VQ, where V is the potential across the capacitor and Q is the charge on its plates. However, this is not correct. As the capacitor is "charging up", initially it is easy to transfer charge from one side to the other. It gets more difficult to transfer charge when there is more charge on the plates.

We can solve the problem in a number of ways. A simple way is to let the charged capacitor discharge through a resistor of resistance R and add up the energy transfered to the resistor. When a capacitor discharges through a resistor, the charge on the plates decreases as $Q_0 e^{-t/(RC)}$. The current is $-dQ/dt$:

$$I = \frac{Q_0}{RC} e^{-t/(RC)} \tag{60}$$

Since the energy transfer rate to the resistor is $I^2 R$, the total energy transfer is:

$$Total\ Energy = \int_0^\infty I^2 R dt \tag{61}$$

Substituting in for the current we have:

$$Total\ Energy = \int_0^\infty \frac{Q^2}{RC^2} e^{-2t/(RC)} dt \tag{62}$$

Carrying out the integral gives

$$Total\ Energy = \frac{Q^2}{2C} \tag{63}$$

This energy, which was transfered to the resistor, was the amount of energy that was stored in the capacitor. One can express this energy in terms of Q and C as above, or in terms of Q and V by substituting $V = Q/C$:

$$
\begin{aligned}
Energy\ in\ Capacitor &= \frac{Q^2}{2C} \\
&= \frac{QV}{2} \\
&= \frac{CV^2}{2}
\end{aligned}
$$

Another interesting way to express the energy stored in a capacitor is in terms of the electric field between the plates. For a parallel plate capacitor the magnitude E of the electric field is $E = Q/(A\epsilon_0)$. Writing Q as $Q = EA\epsilon_0$ and $C = A\epsilon_0/d$ where d is the spacing between the plates, one has:

$$
Energy\ in\ Capacitor = \frac{(A\epsilon_0 E)^2}{2(A\epsilon_0)/d} \tag{64}
$$

Simplifying, we have

$$
Energy\ in\ Capacitor = \frac{\epsilon_0 E^2}{2}\ Ad \tag{65}
$$

The quantity Ad is the volume inside the capacitor. Thus, the quantity $\epsilon_0 E^2/2$ is an energy density which depends only on the electric field. It can shown (using vector calculus) that in general the expression $\epsilon_0 E^2/2$ can be interpreted as the energy density of the electrostatic field. It is often given the symbol U_E:

$$
U_E = \frac{\epsilon_0 E^2}{2} \tag{66}
$$

is the energy density of the electric field. In our upper division course it is shown that this term is part of an energy conservation equation for electromagnetic fields and charges.

Microscopic View of Electric Current

We should consider what is happening at the microscopic level in a wire when current is flowing. If there is an electric field present in the wire, the free electrons will feel a force and accelerate. If the magnitude of the electric field is E, the acceleration of an electron will be $a = eE/m$ where m is the mass of an electron. The electron will keep accelerating until it is stopped by an atom (or another electron) in the conductor. If we denote τ as 1/2 of the average time it takes the electron to travel before it is stopped, the average drift velocity is $v_d \approx eE\tau/m$. If there are n free electrons per volume, then the current density is $J = nev_d$, or

$$\vec{J} = \frac{ne^2\tau}{m}\vec{E} \tag{67}$$

We see that this simple model predicts that $\vec{J}$ is proportional to $\vec{E}$. If τ doesn't depend on the amount of current that is flowing (or temperature), then the material will be Ohmic.

We can get a rough idea of how fast the charge is flowing in a wire. Suppose we have a wire that has a current of one amp and whose diameter is 2 mm. Since $I = JA$, and $J = nev_d$, we have

$$I = nev_dA \tag{68}$$

where n is the number of conduction electrons per m^3 and A is the area. Solving for the drift velocity v_d,

$$v_d = \frac{I}{ne\pi r^2} \tag{69}$$

For copper, $n \approx 10^{28}$ electrons$/m^3$. Using $r = 0.001\ m$ we obtain $v_d \approx 0.2\ mm/sec$.

This (0.2 mm/sec) is very slow. Why does the light go on just as the switch is closed? This is because the electric field in the wire is established very fast, roughly at the speed of light. As soon as this happens electrons all throughout the circuit move simultaneously. The free electrons in the light start moving as soon as the switch is closed. Although the drift speed is slow, the current can be large because there are a lot of electrons moving.