

Supplementary Notes III for Rotational Dynamics

So far we have only considered the motion of "point" objects. We have not allowed the objects to rotate, nor have we considered how forces might produce a "twisting force". In this last section, the last 2 1/2 weeks of the quarter, we will investigate the laws of physics pertaining to rotational motion. As we shall see, we can extend Newton's laws for translational motion to include rotation, without introducing any new "laws of physics".

We will first study the condition for objects not to rotate, then analyze rotations about a fixed axis. Finally, we will extend our ideas to cases in which the axis of rotation can move, while keeping its direction constant.

Torque: twisting force

We all have experience tightening a bolt. If we want it tight, we use a wrench. To apply a large amount of "twisting force", we hold the wrench as far as we can from the bolt and push with a force perpendicular to the wrench arm. The name we call "twisting force" is **torque**. In order to determine the amount of torque, we need to specify three things:

- 1) The axis of rotation. In the case of the bolt, this is where the bolt is.
- 2) The force that is doing the twisting. (The force, direction and magnitude, that you apply to the wrench).
- 3) The position of the force. (How far from the bolt that you apply the force.)

When we speak of torque, we usually say "the torque *about* the axis (or point) O due to the force F applied at the point P ."

All three things matter in determining the torque. We know from experience that a larger force produces a larger torque, and the further away from the axis that the force is applied, the larger the torque, but what is the correct mathematical formula for torque? We will show in class that:

$$\text{torque} = |\vec{r}| \text{ times the component of the force perpendicular to } \vec{r}$$

where \vec{r} is the vector from the axis to the point where the force is applied. The component of the force perpendicular to \vec{r} is the component of the force that does the twisting. The component of the force, \vec{F} , that is parallel to \vec{r} just pushes on the bolt.

This component doesn't cause any twisting about the axis. The symbol for torque used in most physics texts is τ :

$$|\vec{\tau}| = |\vec{r}|F_{\perp} \quad (1)$$

If we let θ be the angle between the vector \vec{r} and the vector \vec{F} , the torque equation can be written as:

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin(\theta) \quad (2)$$

Remember, the vector \vec{r} goes from the axis of rotation to the point at which the force is applied.

Is torque a vector, a scalar, or something else? Torque certainly has a magnitude, which is the amount of twisting force. We can also associate a direction with torque: The axis of rotation of the twist. The axis of rotation is in the direction perpendicular to both the force \vec{F} and the position vector \vec{r} . Thus, we can assign a vector to a particular torque: The magnitude of the vector is the amount of twisting force, $|\vec{r}||\vec{F}|\sin(\theta)$; and the direction is the direction perpendicular to the plane containing \vec{r} and \vec{F} . There is one more decision to make: which direction along the axis of the torque corresponds to a clockwise twist, and which direction to a counter-clockwise twist? Since $\sin(-\theta) = -\sin(\theta)$, the vector for clockwise twist will be in the opposite direction to the one for counter-clockwise twist. The convention that is agreed upon around the world is the following: the direction of the torque vector is the direction that the bolt would move (for normal "right-handed" bolts). Using your right hand, if your fingers curl in the direction of the twist, your thumb points in the direction of the torque.

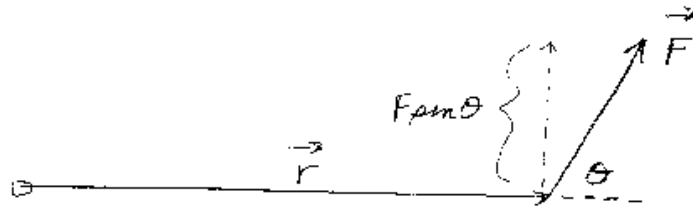
We will show that torques add according to the rules of vector addition, so it is a true vector. In mathematics, there is an operation between vectors that yields another vector. It is the cross product and we can write the torque vector as \vec{r} cross \vec{F} :

$$\vec{\tau} = \vec{r} \times \vec{F} \quad (3)$$

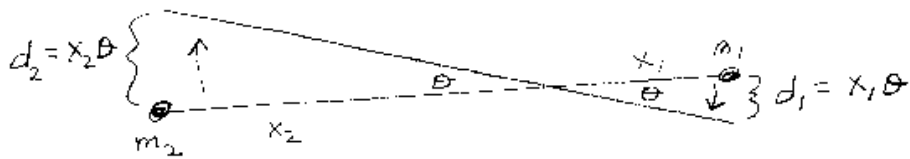
Let's summarize some of the properties of the cross product.

Vector Cross Product

The cross product of vector \vec{A} and \vec{B} is a vector. The magnitude of $\vec{A} \times \vec{B}$ equals the magnitude of \vec{A} , $|\vec{A}|$ times the magnitude of \vec{B} , $|\vec{B}|$, times the sin of the angle between them:



$$|\tau| = |\vec{r}| |\vec{F}| \sin \theta$$



$$W_2 = -m_2 g d_2$$

$$W_1 = m_1 g d_1$$

$$W_g = W_1 + W_2 = \text{change in KE} = 0$$

$$m_1 g d_1 - m_2 g d_2 = 0$$

$$m_1 d_1 = m_2 d_2$$

$$m_1 x_1 \sin \theta = m_2 x_2 \sin \theta$$

$$m_1 x_1 = m_2 x_2$$

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}|\sin(\theta) \quad (4)$$

where the angle θ is the angle between vector \vec{A} and vector \vec{B} . The angle goes from \vec{A} to the vector \vec{B} . The direction of $\vec{A} \times \vec{B}$ is determined by the "right hand" rule described above. Note that since $\sin(-\theta) = -\sin(\theta)$,

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \quad (5)$$

Note that the cross product is maximized when \vec{A} is perpendicular to \vec{B} , and zero when θ is zero or 180° . The unit vectors have simple cross product properties:

$$\begin{aligned} \hat{i} \times \hat{j} &= \hat{k} \\ \hat{j} \times \hat{k} &= \hat{i} \\ \hat{k} \times \hat{i} &= \hat{j} \\ \hat{i} \times \hat{i} &= 0 \\ \hat{j} \times \hat{j} &= 0 \\ \hat{k} \times \hat{k} &= 0 \end{aligned}$$

We can express \vec{A} and \vec{B} in terms of the unit vectors: $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$. If we use the above formulas for the cross products of the unit vectors, after "foil"ing we have:

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} \\ &\quad + (A_xB_y - A_yB_x)\hat{k} \end{aligned}$$

This formula can be succinctly written in terms of the determinant of a matrix:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (6)$$

We have covered all the vector operations that we will need for our first year physics series. Let's summarize them:

Vector Addition: When two vectors are added, the result is another vector. Vector addition is commutative, $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. The "tail-to-tip" method satisfies the

commutative property. In terms of components, $(\vec{A} + \vec{B})_x = A_x + B_x$, $(\vec{A} + \vec{B})_y = A_y + B_y$, and $(\vec{A} + \vec{B})_z = A_z + B_z$. If physical quantities are vectors, addition of vectors is used to find the NET sum of the quantities (e.g. the net force).

Multiplication of a scalar and a Vector: Multiplication of a vector by a scalar c changes the magnitude of the vector by the factor $|c|$. If $c > 0$, then the direction remains the same, but if $c < 0$ the direction is reversed. Multiplication by a scalar enables one to define unit vectors, whose linear combinations span the vector space.

Scalar product of two vectors: The scalar product of one vector with another vector yields a scalar. Since two vectors are involved, the scalar product must depend on the magnitudes of each vector and the relative direction of one to the other. If θ is the angle between the vectors, the scalar product could depend on either $\sin(\theta)$ or $\cos(\theta)$. Since the scalar product should be commutative, $\cos(\theta)$ is the only choice since $\cos(-\theta) = \cos(\theta)$. Thus in order to be commutative, the scalar product must be $\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos(\theta)$. In terms of components, $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$. In physics, the scalar product represents the component of one vector in the direction of another (times its magnitude). It allowed us to express the work done by a force in acting in a certain direction easily ($\vec{F} \cdot \Delta\vec{r}$).

Vector product of two vectors: The vector product of one vector with another yields a vector. Since two vectors are involved here also, the vector product must depend on the magnitudes of each vector and the relative direction of one to the other. Since the result must be a vector, there is only one special direction: the direction perpendicular to \vec{A} and \vec{B} . Should $\sin(\theta)$ or $\cos(\theta)$ be used? Theta equal to zero helps us determine the proper choice. If \vec{A} is parallel to \vec{B} then there is no special direction, so the vector product must be zero in this case. The trig function that is zero for $\theta = 0$ is $\sin(\theta)$. Thus, the only way to define a vector product of two vectors is to use the sin function as described above.

These vector operations will be very important in describing the electro-magnetic interaction. This is because the magnetic part of the interaction depends on the velocity of the particles. With so many vectors involved, \vec{F} , \vec{r} , and \vec{v} , the scalar and vector products are important. It will also be necessary to consider derivatives of vectors, i.e. vector calculus. Here we will consider rotational motion, and we will see the usefulness of the vector product.

Statics

In the first part of the course, we showed that in order for an object to stay at rest, the vector sum of the forces on the object must be zero. This is all one needs if the object is a point particle. However, if the object has size, the location of the forces on the object is important. Eventhough the sum of the forces is zero, the object might rotate about its center-of-mass. In order for the object to stay stationary and not rotate, the sum of the torques must be zero about any axis. These are the two conditions for an object to remain "static", i.e. not translate and not rotate:

$$\begin{aligned}\sum_i \vec{F}_i &= 0 \\ \sum_i \vec{\tau}_i &= 0\end{aligned}$$

where the torque sum can be taken about any axis.

In class we will do many examples of many objects in static equilibrium. In evaluating the torque sum, a proper choice of axis can reduce the mathematical complexity.

You might wonder why the twisting force equals the product of force times distance from the axis. Consider a simple rigid object consisting of two masses connected by a massless rod on the earth. Suppose we consider balancing the object about a point on the massless rod. Let the object to the right have a mass m_1 and be at a distance x_1 from the axis, and let the object to the left have a mass of m_2 and be at a distance of x_2 from the axis. If the axis is the balance point then the rod will rotate with constant angular speed if slightly pushed. To see how the speed might change, we can use the work-energy theorem. Let θ be the angle that the rod rotates. Then m_1 moves a distance $x_1\theta$, and m_2 moves a distance $x_2\theta$. The work done by gravity on the mass m_1 is $m_1gx_1\theta$, and the work done by gravity on the mass m_2 is $-m_2gx_2\theta$. Both objects sweep out the same angle θ . If the rod is to rotate with constant speed, then the Net Work done by gravity must be zero:

$$m_1gx_1\theta - m_2gx_2\theta = 0 \tag{7}$$

This equations yields $m_1gx_1 = m_2gx_2$. Force times distance on one side equals force times distance on the other when the rod is balanced. We also verified this condition for balance in class. If m_1x_1 were to be greater then m_2x_2 then gravity would do more positive work on m_1 then negative work on m_2 because the distance traveled by m_1 would be greater than m_2/m_1 times x_2 . There would be net work done by gravity and the rod's "angular" speed would increase. We will derive this result more generally using the vector cross product later.

Rotational Dynamics

We will first discuss the dynamics of rotational motion about a fixed axis, then treat the case when the axis of rotation can translate.

Rotation of a Rigid Object about a fixed axis

What is meant by a rigid object (or rigid body) is an object for which the separation between all pairs of points remains constant. When we talk about an object in this section, we mean rigid object. Let's first discuss how we are going to describe rotation about an axis.

Describing rotations about a fixed axis

If the axis of rotation is fixed, then the objects orientation can be specified by an angle (with respect to a reference direction). Specifying the angle in units of degrees is an earthly standard. The assignment of 360° for a complete rotation is probably because the earth turns 365 times about its axis for every revolution around the sun. One degree is approximately the angle that the earth "sweeps out" around the sun in one day. A more universal measure of angle is the radian. The angle in radians is determined by drawing an arc of radius r and measuring the distance of the arc, s , that subtends the angle. The angle θ in radians is defined as $\theta \equiv s/r$. This definition gives the same value for θ for any radius value r . This is really a universal definition which could be agreed upon by people living on any planet.

You might be wondering what kind of quantity angular displacement is. Is it a scalar or a vector, or something else? Angular displacement has a magnitude, but is there a direction associated with the displacement? There is an axis of rotation and a distinction can be made versus clockwise and counter-clockwise displacements. Thus, a direction and magnitude can be associated with an angular displacement: the direction is along the axis of rotation, with the right-hand rule defining positive displacements; the magnitude equals the amount of angular displacement. Although angular displacements have a magnitude and a direction, angular displacements do not add according to the laws of vector addition. Try it by rotating a book with the following example: $(\pi/2)\hat{i} + (\pi/2)\hat{j}$ does not equal $(\pi/2)\hat{j} + (\pi/2)\hat{i}$. Actually, $(\pi/2)\hat{i} + (\pi/2)\hat{j} = (-\pi/2)\hat{k} + (\pi/2)\hat{i}$. Rotations are not vectors, they form what mathematicians call a group.

An objects motion about a fixed axis is one of spinning. We can quantify how fast an object is spinning by how fast the angle θ is changing in time. We refer to

the spinning speed as the **angular velocity**, and give it the symbol ω (omega). The magnitude of the instantaneous angular velocity is the magnitude of the instantaneous time rate of change of θ :

$$|\omega| \equiv \left| \frac{d\theta}{dt} \right| \quad (8)$$

Is the angular velocity a scalar or a vector? As with displacement it has a magnitude and a direction. The direction is the axis of rotation, with clockwise and counter-clockwise determined by the right-hand rule. Curl the fingers of your right hand in the direction of the rotation, and your thumb points in the direction or $\vec{\omega}$. As with angular displacement, angular velocities do not add like vectors. However it is still useful to associate a vector with angular velocity. The usefulness arises when one wants to relate the velocity of a particle rotating about an axis.

Consider an object rotating about an axis with angular velocity $|\vec{\omega}|$. Let the origin of your coordinate system be located on the axis of rotation. That is, the vector $\vec{\omega}$ goes through the origin. Define the vector \vec{r} as the position vector to the object that is rotating. The speed of the object is $|\vec{\omega}|$ times the perpendicular distance of the object to $\vec{\omega}$: $|\vec{v}| = |\vec{\omega}| |\vec{r}| \sin(\theta)$, where θ is the angle between $\vec{\omega}$ and \vec{r} . The direction of the particles velocity is perpendicular to both \vec{r} and the vector $\vec{\omega}$. Thus, the velocity \vec{v} of a particle at a position \vec{r} can be written as

$$\vec{v} = \vec{\omega} \times \vec{r} \quad (9)$$

This is a succinct way of expressing the velocity of an object that is rotating, and will be useful in deriving the equations of rotational motion.

Objects don't always rotate with a constant angular velocity. As was done for the case of translational motion, acceleration is defined as the change in velocity. For the case of rotation, we can define the angular acceleration as the change in angular velocity.

$$|\alpha| \equiv \left| \frac{d\omega}{dt} \right| \quad (10)$$

In this class we will only consider situations in which the axis of rotation does not change direction. Thus, α will be in the same direction as ω .

For the special case of **constant angular acceleration** the relationship between the angular velocity ω and angular displacement θ is simple. Since θ , ω , and α have the same mathematical relationship to each other as x , v , and a for one dimensional motion, the formulas for constant angular acceleration are the same as those for constant acceleration:

$$\begin{aligned}
\alpha &= \alpha_0 \\
\omega &= \alpha_0 t + \omega_0 \\
\theta &= \frac{\alpha_0}{2} t^2 + \omega_0 t + \theta_0
\end{aligned}$$

where ω_0 is the initial angular velocity, and θ_0 is the initial angular displacement. The angular acceleration is α_0 , which is constant. From the middle equation, $t = (\omega - \omega_0)/\alpha_0$. This expression for t can be substituted into the last equation to give:

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \quad (11)$$

Remember, the above equations are only valid for the special case of *constant* angular acceleration.

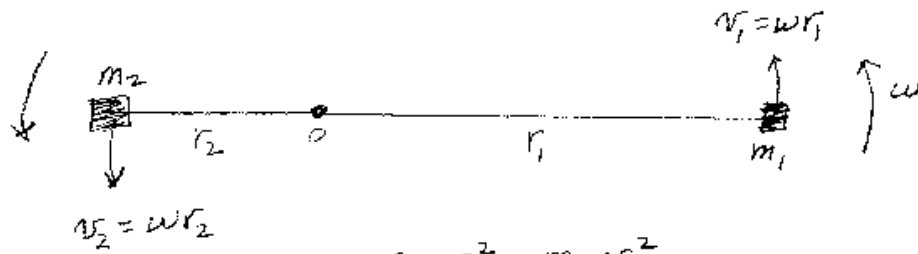
Energy of Rotation for a Rigid Object about a fixed axis

When we analyzed translational motion, we started first with the relationship between acceleration, force and mass. Then we discussed the work-energy theorem and kinetic energy. For rotational motion, it is easier to first discuss energy considerations, then how torque changes angular velocity.

Let's consider the energy of motion (rotational kinetic energy) of a rigid object rotating about a fixed axis. We first start with the simple case of a rigid object consisting of two objects of mass m_1 and m_2 connected by a massless rod. Let the axis of rotation be on the axis, and mass m_1 be located a distance of r_1 from the axis and mass m_2 be located a distance of r_2 . If the rod is rotating about its axis with an angular velocity ω , the speed v_i of each object is $v_i = r_i\omega$. The energy of motion of the rigid object is

$$\begin{aligned}
K.E. &= \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 \\
&= \frac{m_1}{2} (\omega r_1)^2 + \frac{m_2}{2} (\omega r_2)^2 \\
K.E. &= \frac{m_1 r_1^2 + m_2 r_2^2}{2} \omega^2
\end{aligned}$$

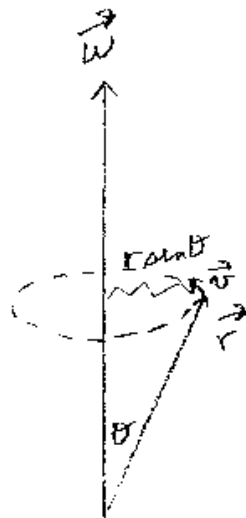
A nice property of rotational motion about a fixed axis is that the angular velocity ω is the same for both objects. This allows a factorization of the kinematical quantity (ω) from the inertial quantity $m_1 r_1^2 + m_2 r_2^2$. As we shall see, the combination of masses



$$K.E. = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2$$

$$K.E. = \frac{m_1}{2} (\omega r_1)^2 + \frac{m_2}{2} (\omega r_2)^2$$

$$K.E. = \frac{(m_1 r_1^2 + m_2 r_2^2)}{2} \omega^2$$



$$|\vec{v}| = (r \sin \theta) \omega$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

times distance squared always appears as an inertia to changes in rotational motion. It is a special combination, and deserves a special name, **rotational inertial**, and is given the symbol I_A :

$$I_A \equiv m_1 r_1^2 + m_2 r_2^2 \quad (12)$$

where A denotes the axis about which the rigid object is rotating. The rotational inertial depends not only on the amount of mass that is rotating, but also on the location of the masses (i.e. the distance the masses are from the axis of rotation). When dealing with rotational inertial, remember to always specify the axis about which the inertial is determined.

The expression for rotational inertial can be generalized to any number of point particles that are "rigidly" connected. If N point objects are connected so that their relative distances do not change, the rotational inertial about the axis A is

$$\begin{aligned} I_A &= m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \dots \\ &= \sum_{i=1}^N m_i r_i^2 \end{aligned}$$

This expression can also be extended to a solid object. To do this, we divide the solid object up into N small pieces. We can denote m_i as the mass of piece i , and r_i as the distance to the axis of rotation. Note: the distance r_i is the shortest distance from m_i to the axis (i.e. the perpendicular distance). The mass m_i can be written as $\rho \Delta V_i$. If we take the limit as $N \rightarrow \infty$ and $\Delta V_i \rightarrow 0$, the sum becomes an integral:

$$I_A = \int \rho r^2 dV \quad (13)$$

In lecture we will calculate I_A for a thin rod of length L and mass M for axis of rotation A about an end of the rod and about the middle of the rod. We will also calculate I_A for a uniform cylinder for rotations about the center.

The kinetic energy of rotation (rotational kinetic energy) for any rigid object rotating about the *fixed axis* A is

$$K.E.(of\ rotation) = \frac{I_A}{2} \omega_A^2 \quad (14)$$

where I_A is the rotational inertia about the axis A , and ω_A is the angular velocity about the axis A .

Dynamics of a rigid object about a fixed axis

Many engineering applications involve rigid objects rotating about fixed axes. For this reason we discuss briefly this special situation. The simplest case is one particle of mass m attached by massless rod to an axis point A . Let the position vector from the axis to the particle be \vec{r} . The particle is confined to rotate a distance $r \equiv |\vec{r}|$ away from the point A . Suppose the particle is subject to the net force \vec{F}_{net} at a particular time. What kind of motion will happen? Only the component of the force \vec{F}_{net} perpendicular to \vec{r} will cause any change in the particles speed. The component perpendicular to \vec{r} is $|\vec{F}_{net}|\sin(\theta)$ where θ is the angle between \vec{r} and \vec{F}_{net} . This component of the force will cause an acceleration equal to $|\vec{F}_{net}|\sin(\theta)/m$ along the path of the rotating particle.

$$|\vec{F}_{net}|\sin(\theta) = m|\vec{a}| \quad (15)$$

If we multiply both sides by r , we have

$$r|\vec{F}_{net}|\sin(\theta) = mr|\vec{a}| \quad (16)$$

Since the particle is always the same distance from A , the particle's acceleration is along the circle confining the particle. Thus, the acceleration can be specified by the angular acceleration about the axis point A , $|\vec{a}| = |\vec{\alpha}|r$

$$\begin{aligned} r|\vec{F}_{net}|\sin(\theta) &= mr^2\frac{|\vec{a}|}{r} \\ &= mr^2|\vec{\alpha}| \end{aligned}$$

The left side of the equation is recognized as the torque about the axis A . Since $\vec{\alpha}$ will point along the same direction as $\vec{r} \times \vec{F}_{net}$,

$$\begin{aligned} \vec{r} \times \vec{F}_{net} &= mr^2\vec{\alpha}_A \\ \tau_{net} &= I_A\alpha_A \end{aligned}$$

where the subscript A refers to the axis A : I and $\vec{\alpha}$ are measured about the axis A . From this derivation, one can see that the end result is just Newton's second law for translational motion applied to the motion in a circle that a particle must move if connected by a rigid (massless) rod to an axis point. As we will show, this expression also applies in general to a solid rigid object rotating about a fixed axis.

Consider a rigid object confined to rotate about a fixed axis A . Suppose a force \vec{F} is applied at a position \vec{r} from the axis. Let $\Delta\theta$ be the (small) angle the object turns while the force is applied. The work done on the system W by the force is the component of the force perpendicular to \vec{r} times $r\Delta\theta$, which is the distance the force acts:

$$W = F_{\perp}(\Delta\theta r) \quad (17)$$

where F_{\perp} is the component of \vec{F} perpendicular to \vec{r} . This can also be written as

$$W = |\vec{r} \times \vec{F}|\Delta\theta \quad (18)$$

The torque times the angular displacement is the work that the torque does. If this is the only torque that does work on the system, then it is equal to a change in kinetic energy (work-energy theorem):

Torque times angular displacement equals change in K.E.

Letting $\Delta(K.E.)$ denote the change in kinetic energy, we have

$$\begin{aligned} \tau_A \Delta\theta &= \Delta(K.E.) \\ &= \Delta\left(\frac{I_A}{2}\omega_A^2\right) \end{aligned}$$

If the torque occurs over a small time interval Δt , the equation becomes

$$\tau_A \frac{\Delta\theta}{\Delta t} = \frac{\Delta\left(\frac{I_A}{2}\omega_A^2\right)}{\Delta t} \quad (19)$$

Taking the limit as Δt goes to zero results in derivatives with respect to t :

$$\tau_A \frac{d\theta}{dt} = \frac{I_A}{2} \frac{d\omega^2}{dt} \quad (20)$$

or

$$\tau_A \omega = I_A \omega \frac{d\omega}{dt} \quad (21)$$

canceling the ω on both sides:

$$\tau_A = I_A \alpha_A \quad (22)$$

Thus, for a rigid object confined to rotate about a fixed axis, **the net torque about the axis equals the rotational inertial times the angular acceleration about the axis**. To generalize to situations where the axis of rotation is not fixed and where the rotational inertial can change, it is useful to introduce a new quantity: angular momentum.

Rotational Dynamics if the axis of rotation is moving

Kinetic Energy for objects rotating and translating

Before we discuss how torques affect rotational motion, it is instructive to consider the energy of motion for objects that are rotating and translating (moving). The complicated situation of rotation while the object is moving is made simple by expressing 'translation of' and 'rotation about' the center-of-mass of the object. The expression for the total energy of motion is

$$Total\ K.E. = \frac{M_{total}}{2} V_{c.m.}^2 + \frac{I_{c.m.}}{2} \omega_{c.m.}^2 \quad (23)$$

Let's derive this equation to understand why the center-of-mass plays such an important role. Consider a system of particles, or a rigid object, that is subject to certain forces. We will analyze the system of particles in an inertial reference (lab frame) in which the center of mass (c.m.) of the system is moving. The velocity of the c.m. for N particles is given by

$$\vec{V}_{c.m.} = \frac{\sum_{i=1}^N m_i \vec{v}_i}{M_{tot}} \quad (24)$$

where $M_{tot} = \sum_{i=1}^N m_i$ is the total mass of the system. Let \vec{u}_i be the velocity of object i with respect to the center-of-mass reference frame. That is:

$$\vec{v}_i = \vec{V}_{c.m.} + \vec{u}_i \quad (25)$$

where \vec{v}_i is the velocity of particle i as measured in the inertial reference frame (lab frame), and $\vec{V}_{c.m.}$ is the velocity of the c.m. Note that in the c.m. frame:

$$\sum_{i=1}^N m_i \vec{u}_i = 0 \quad (26)$$

The center of mass frame is also called the center of momentum frame, since the total momentum of the system is zero in the c.m. frame.

The total kinetic energy is:

$$\begin{aligned} \text{Total } K.E. &= \sum_{i=1}^N \frac{m_i}{2} v_i^2 \\ &= \sum_{i=1}^N \frac{m_i}{2} (\vec{v}_i \cdot \vec{v}_i) \end{aligned}$$

Substituting the expression for \vec{v}_i above gives

$$\begin{aligned} \text{Total } K.E. &= \sum_{i=1}^N \frac{m_i}{2} (\vec{V}_{c.m.} + \vec{u}_i) \cdot (\vec{V}_{c.m.} + \vec{u}_i) \\ &= \sum_{i=1}^N \frac{m_i}{2} (V_{c.m.}^2 + 2\vec{V}_{c.m.} \cdot \vec{u}_i + u_i^2) \end{aligned}$$

In the center of mass frame, the middle term sums to zero:

$$\begin{aligned} \sum_{i=1}^N m_i \vec{V}_{c.m.} \cdot \vec{u}_i &= \vec{V}_{c.m.} \cdot \sum_{i=1}^N m_i \vec{u}_i \\ &= 0 \end{aligned}$$

since in the c.m. frame $\sum m_i \vec{u}_i = 0$. Thus, the total K.E. separates into two parts:

$$\begin{aligned} \text{Total } K.E. &= \sum_{i=1}^N \frac{m_i}{2} (V_{c.m.}^2 + u_i^2) \\ &= \frac{M_{total}}{2} V_{c.m.}^2 + \frac{1}{2} \sum_{i=1}^N m_i u_i^2 \end{aligned}$$

The second term in the expression is the kinetic energy as measured in the c.m. frame. For the case of a rigid object, $\vec{u}_i = \vec{\omega} \times \vec{r}_i$. The magnitude of \vec{u}_i squared is $(\vec{\omega} \times \vec{r}_i)^2$, which after summing over all particles (and some algebra) gives $(\sum m_i r_i^2) \omega_{c.m.}^2$. For a rigid object, the above expression becomes

$$\text{Total } K.E. = \frac{M_{total}}{2} V_{c.m.}^2 + \frac{I_{c.m.}}{2} \omega_{c.m.}^2 \quad (27)$$

Thus, **the total energy of motion equals the translational kinetic energy of the center of mass plus the rotational kinetic energy about the center of mass.** The key for the separation of the translational energy of the system from the rotational energy about an axis was to find a reference frame in which $\sum m_i u_i = 0$. This is the definition of the center-of-mass (or momentum) frame. A complicated expression is made simple when viewed from the right perspective.

We would like to generalize the result for rigid object rotating about a fixed axis to the case of an axis of rotation that is not fixed as well as the case when I can change. We will consider a system made up of N particles. If the system is a rigid body, then the distances between the particles is fixed and N approaches infinity as the particle size goes to zero. Let \vec{r}_i be the position vector, \vec{v}_i be the velocity, and m_i be the mass of particle i . Note that \vec{r}_i and \vec{v}_i both depend on time. The key to a succinct formalism is the following vector identity which applied to every particle:

$$\frac{d(\vec{r} \times \vec{p})}{dt} = \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) \quad (28)$$

where \vec{p} is the momentum of the particle. For a single particle, the momentum is $\vec{p} = m\vec{v}$. So, the first term on the right hand side is

$$\begin{aligned} \frac{d\vec{r}}{dt} \times \vec{p} &= \vec{v} \times (m\vec{v}) \\ &= m(\vec{v} \times \vec{v}) \\ &= 0 \end{aligned}$$

Thus, we have

$$\frac{d(\vec{r}_i \times \vec{p}_i)}{dt} = \vec{r}_i \times \frac{d\vec{p}_i}{dt} \quad (29)$$

for each particle i . This is a wonderful vector equation. Now we can put in the physics: $\vec{F}_{i-net} = (d\vec{p}_i)/(dt)$, where \vec{F}_{i-net} is the net force on particle i . Substituting gives:

$$\frac{d(\vec{r}_i \times \vec{p}_i)}{dt} = \vec{r}_i \times \vec{F}_{i-net} \quad (30)$$

The expression on the right side is recognized as the net torque on particle i about the origin, τ_{i-net} :

$$\frac{d(\vec{r}_i \times \vec{p}_i)}{dt} = \tau_{i-net} \quad (31)$$

Thus, the torque on a particle changes the quantity $\vec{r} \times \vec{p}$. What is $\vec{r} \times \vec{p}$? It's magnitude is $|\vec{r}||\vec{p}|\sin\theta$, where θ is the angle between \vec{r} and \vec{p} . $|\vec{p}|\sin(\theta)$ is the component of momentum perpendicular to \vec{r} . It is in essence the "rotational" momentum of the particle about the origin. The name given to $\vec{r} \times \vec{p}$ is the **angular momentum** of the particle about the origin. Note that *the particle does not have to be rotating to have angular momentum*. A particle moving in a straight line can have an angular momentum about a point (or origin).

Angular Momentum

The symbol for angular momentum is usually \vec{L} . For particle "i", $\vec{L}_i \equiv \vec{r}_i \times \vec{p}_i$. Note that the angular momentum of a particle will depend on the frame of reference, since the position vector \vec{r} is measured from the origin of a particular reference frame. For a single particle, the above equation says that the time rate of change of a particles angular momentum equals the net torque on the particle.

$$\frac{d\vec{L}_i}{dt} = \tau_{i-net} \quad (32)$$

where the angular momentum and the torque are evaluated about the same axis. Note, that if the net torque on a particle equals zero, then the angular momentum of the particle about the axis is a constant of the motion. Forces that do not produce a torque are forces that lie along the direction \vec{r} .

For a collection of particles, we can define the total angular momentum of the system as the vector sum of the angular momentum of the particles:

$$\vec{L}_{tot} \equiv \sum_{i=1}^N \vec{r}_i \times \vec{p}_i \quad (33)$$

Since \vec{r} and \vec{p} are true vectors, the angular momentum \vec{L} and \vec{L}_i will also have the properties of a vector. Note that \vec{L}_{tot} will depend on the choice of origin.

Rotational dynamics is most easily handled by considering the time rate of change of the total angular momentum:

$$\frac{d\vec{L}_{tot}}{dt} = \sum_{i=1}^N \frac{d(\vec{r}_i \times \vec{p}_i)}{dt}$$

$$= \sum_{i=1}^N \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

The first term on the right side equals zero as before:

$$\begin{aligned} \frac{d\vec{r}_i}{dt} \times \vec{p}_i &= \vec{v}_i \times \vec{p}_i \\ &= \vec{v}_i \times m_i \vec{v}_i \\ &= 0 \end{aligned}$$

since $\vec{v}_i \times \vec{v}_i$ equals zero. Thus, we have the same result for a system of particles as we did with one particle:

$$\begin{aligned} \frac{d\vec{L}_{tot}}{dt} &= \sum_{i=1}^N \vec{r}_i \times \frac{d\vec{p}_i}{dt} \\ &= \sum_{i=1}^N \vec{r}_i \times \vec{F}_{i-net} \\ &= \vec{\tau}_{net} \end{aligned}$$

The time rate of change of the total angular momentum of a system of particles equals the net torque on the system. Note that this equation is valid for any axis point about which the torque and angular momentum are calculated (or measured).

For a system of particles, the above equation can be somewhat simplified. One can separate the force on each particle into a part that comes from other particles *in the system* or internal forces, \vec{F}_{i-int} , and another part originating from sources *outside the system* or external forces, \vec{F}_{i-ext} :

$$\vec{F}_{i-net} = \vec{F}_{i-int} + \vec{F}_{i-ext} \quad (34)$$

The internal forces are forces between the particles in the system. Newton's third law states that $\vec{F}_{ij} = -\vec{F}_{ji}$. If these action-reaction forces act along the line joining particle i and j , then the sum of the torques due to the internal forces will add to zero. Thus, the net torque on the system, $\vec{\tau}_{net}$, will equal the sum of the torques due to external forces, or net external torque, $\vec{\tau}_{net-ext}$:

$$\frac{d\vec{L}_{tot}}{dt} = \vec{\tau}_{net-ext} \quad (35)$$

This equation has an important consequence for a system of particles. **If the net torque on a system is zero, then the total angular momentum is conserved!** We can add another conserved quantity to our list: total momentum of a system, total angular momentum of a system, mechanical energy for conservative forces. The symmetry responsible for the conservation of angular momentum is rotational symmetry. Angular momentum and its conservation play a very important role in our understanding of atomic, nuclear and particle physics.

The above equation is quite general. In this class we will consider two special cases: 1) rotation of a rigid body about a fixed axis, and 2) the case where the angular momentum vector does not change direction (although the axis of rotation can translate). These cases are important in engineering applications.

Angular momentum of a rigid object about a fixed axis

The expression for the angular momentum of a rigid object rotating about a fixed axis takes on a simple form. Consider a flat rigid object that is confined to rotate about an axis perpendicular to its surface. Divide the object up into N small pieces. Let the mass of piece i be m_i , and let the mass be located a distance r_i from the axis. Let the angular velocity about the axis be ω . The speed of each piece is $\omega|r_i|$, and the direction of the velocity is perpendicular to \vec{r}_i . The angular momentum will lie along the direction of $\vec{\omega}$ with magnitude:

$$\begin{aligned} |\vec{L}| &= \sum_{i=1}^N r_i m_i v_i \\ &= \sum_{i=1}^N r_i m_i \omega r_i \\ &= \left(\sum_{i=1}^N m_i r_i^2 \right) \omega \\ |\vec{L}| &= I \omega \end{aligned}$$

where the direction of \vec{l} is in the same direction as $\vec{\omega}$, and all quantities are calculated about the same axis, A . Written more precisely:

$$\vec{L}_A = I_A \vec{\omega}_A \tag{36}$$

The relationship between torque and angular momentum is

$$\begin{aligned}
\vec{\tau}_{A-net} &= \frac{d\vec{L}_A}{dt} \\
&= I_A \frac{d\vec{\omega}_A}{dt} \\
\vec{\tau}_{A-net} &= I_A \vec{\alpha}_A
\end{aligned}$$

where $\vec{\tau}_{A-net}$ is the net torque about the axis A . You might remember that this is the same result we obtained before using energy considerations: the net work done by the torque equals the change in the rotational energy of the object.

Translation and rotation of a system of particles

The translational motion and rotational motion of a rigid object (or a system of particles) separate if the center-of-mass is used as a reference point. The position of the center of mass for a system of N particles, \vec{R}_{cm} is:

$$\vec{R}_{cm} \equiv \frac{\sum_{i=1}^N m_i \vec{r}_i}{\sum_{i=1}^N m_i} \quad (37)$$

Letting $M_{tot} \equiv \sum_{i=1}^N m_i$, we can also write the center of mass position as

$$\vec{R}_{cm} \equiv \frac{\sum_{i=1}^N m_i \vec{r}_i}{M_{tot}} \quad (38)$$

The velocity of the center of mass, \vec{V}_{cm} is found by differentiating the expression for \vec{R}_{cm} with respect to time:

$$\vec{V}_{cm} \equiv \frac{\sum_{i=1}^N m_i \vec{v}_i}{M_{tot}} \quad (39)$$

The center of mass is a special point for a system of particles, or a rigid object. For a reference frame that *moves such that the center of mass is at the origin*, we have $\sum_{i=1}^N m_i \vec{r}_i = 0$ and $\sum_{i=1}^N m_i \vec{v}_i = 0$. This reference frame is called the "center-of-mass" frame, or "center-of-momentum" frame. By analyzing the dynamics from this frame, the translational motion of the center-of-mass of a system (or rigid object) separates from the rotational motion about the center-of-mass. This can be seen as follows:

Consider a frame of reference at rest with respect to the earth. We will refer to this frame of reference as the lab frame. Let \vec{R}_i and \vec{V}_i be the position and the velocity of

particle i as measured by the lab frame. Let \vec{r}_i and \vec{v}_i be the position and velocity of particle i as measured in the center of mass frame described above. If \vec{R}_{cm} and \vec{V}_{cm} are the position and velocity of the center of mass of the system (or rigid object), we have

$$\vec{R}_i = \vec{R}_{cm} + \vec{r}_i \quad (40)$$

for the position of particle i , and

$$\vec{V}_i = \vec{V}_{cm} + \vec{v}_i \quad (41)$$

for the velocity of particle i . The angular momentum of the system of particles in the lab reference frame is:

$$\vec{L}_{lab} = \sum_{i=1}^N \vec{R}_i \times m_i \vec{V}_i \quad (42)$$

where m_i is the mass of particle i , and we have written the momentum as mass times velocity, $m_i \vec{V}_i$. If we express these variables in terms of the position and velocity as measured in the center of mass frame, we have

$$\vec{L}_{lab} = \sum_{i=1}^N (\vec{R}_{cm} + \vec{r}_i) \times m_i (\vec{V}_{cm} + \vec{v}_i) \quad (43)$$

After expanding the product on the right side of the equation, we have four terms:

$$\begin{aligned} \vec{L}_{lab} &= \sum_{i=1}^N (\vec{R}_{cm} \times m_i \vec{V}_{cm}) \\ &\quad + \sum_{i=1}^N (\vec{R}_{cm} \times m_i \vec{v}_i) \\ &\quad + \sum_{i=1}^N (\vec{r}_i \times m_i \vec{V}_{cm}) \\ &\quad + \sum_{i=1}^N (\vec{r}_i \times m_i \vec{v}_i) \end{aligned}$$

However, the middle two terms are zero due to the properties of the center of mass reference frame. This can be seen as follows. The second term is

$$\begin{aligned}\sum_{i=1}^N (\vec{R}_{cm} \times m_i \vec{v}_i) &= \vec{R}_{cm} \times \sum_{i=1}^N m_i \vec{v}_i \\ &= 0\end{aligned}$$

since $\sum_{i=1}^N m_i \vec{v}_i$ is zero in the center of mass frame. The third term is also zero,

$$\begin{aligned}\sum_{i=1}^N (\vec{r}_i \times m_i \vec{V}_{cm}) &= \left(\sum_{i=1}^N m_i \vec{r}_i \right) \times \vec{V}_{cm} \\ &= 0\end{aligned}$$

since $\sum_{i=1}^N m_i \vec{r}_i$ is zero in the center of mass frame. So the angular momentum measured in the lab frame separates into two pieces:

$$\begin{aligned}\vec{L}_{lab} &= \sum_{i=1}^N (\vec{R}_{cm} \times m_i \vec{V}_{cm}) + \sum_{i=1}^N (\vec{r}_i \times m_i \vec{v}_i) \\ &= (\vec{R}_{cm} \times m_{tot} \vec{V}_{cm}) + \sum_{i=1}^N (\vec{r}_i \times m_i \vec{v}_i)\end{aligned}$$

The first term on the right is the angular momentum as if all the mass were at the center of mass point, and the second term is the angular momentum about the center of mass.

The torque in the lab frame is

$$\begin{aligned}\vec{\tau}_{lab} &= \sum_{i=1}^N \vec{R}_i \times \vec{F}_i \\ &= \sum_{i=1}^N (\vec{R}_{cm} + \vec{r}_i) \times \vec{F}_i \\ &= (\vec{R}_{cm} \times \sum_{i=1}^N \vec{F}_i) + \left(\sum_{i=1}^N \vec{r}_i \times \vec{F}_i \right) \\ &= (\vec{R}_{cm} \times \vec{F}_{net}) + \left(\sum_{i=1}^N \vec{r}_i \times \vec{F}_i \right)\end{aligned}$$

We showed that Newton's laws of motion applied to rotations gives $\vec{\tau}_{lab} = d\vec{L}_{lab}/dt$. Equating the two expressions above gives:

$$\begin{aligned}
(\vec{R}_{cm} \times \vec{F}_{net}) + \left(\sum_{i=1}^N \vec{r}_i \times \vec{F}_i\right) &= \frac{(d(\vec{R}_{cm} \times m_{tot}\vec{V}_{cm}) + \sum_{i=1}^N (\vec{r}_i \times m_i\vec{v}_i))}{dt} \\
&= \vec{R}_{cm} \times \frac{dm_{tot}\vec{V}_{cm}}{dt} + \sum_{i=1}^N \frac{(\vec{r}_i \times m_i\vec{v}_i)}{dt} \\
&= (\vec{R}_{cm} \times \vec{F}_{net}) + \sum_{i=1}^N \frac{(\vec{r}_i \times m_i\vec{v}_i)}{dt} \\
\sum_{i=1}^N \vec{r}_i \times \vec{F}_i &= \sum_{i=1}^N \frac{(\vec{r}_i \times m_i\vec{v}_i)}{dt} \\
\vec{\tau}_{c.m.} &= \frac{d\vec{L}_{c.m.}}{dt}
\end{aligned}$$

Thus, even though the center of mass is moving, the net torque about the center of mass equals the time rate of change of the angular momentum about the center of mass. This is a very nice result: analyzing the motion about the center of mass greatly simplifies our analysis.

Summary of Rotational Dynamics

This concludes the discussion on rotational motion for our first year physics course. The relationships that we developed on rotational dynamics were derived from Newton's Laws of motion. We did not need to introduce any "new physics" the past 3 weeks. We applied the laws for translation motion to rotations. We realized that the rotational part of the motion is the part perpendicular to the position vector, \vec{r} . If \vec{A} is a translational quantity, then $|\vec{r}||\vec{A}|\sin\theta$ is the corresponding rotational quantity. Here \vec{A} can be the force, momentum, velocity or acceleration vectors, and θ is the angle between \vec{r} and \vec{A} . The vector cross product is the mathematical operation that facilitates our analysis. Thus starting with Newton's second law for particle i in a system of particles:

$$\vec{F}_{i-net} = \frac{d\vec{p}_i}{dt} \tag{44}$$

we can take the vector product of both sides with \vec{r}_i :

$$\vec{r}_i \times \vec{F}_{i-net} = \vec{r}_i \times \frac{d\vec{p}_i}{dt}$$

$$= \frac{d(\vec{r}_i \times \vec{p}_i)}{dt}$$

where the second equation equals the first since $\vec{v}_i \times m_i \vec{v}_i = 0$. Thus, we are motivated to define a "twisting force" vector (torque) as $\vec{r} \times \vec{F}$, and a "rotational momentum" vector (angular momentum) as $\vec{r} \times \vec{p}$. With these definitions, we have:

$$\vec{\tau}_{i-net} = \frac{d\vec{L}_i}{dt} \tag{45}$$

Summing over the N particles in a system gives:

$$\begin{aligned} \sum_{i=1}^N \vec{\tau}_{i-net} &= \sum_{i=1}^N \frac{d\vec{L}_i}{dt} \\ \vec{\tau}_{net} &= \frac{d\vec{L}_{net}}{dt} \end{aligned}$$

From this equation, we showed how translation and rotation can be separated from each other via the center-of-mass reference frame.