Supplimentary Notes II for Newtonian Mechanics

In the first four weeks of the course we discussed Newton's laws of motion, in which the approach taken is one of forces and motion. We discussed three experiments that lead to the relationship between force and motion: $\vec{F}_{Net} = m\vec{a}$, and the "symmetry" of interactions: $\vec{F}_{12} = -\vec{F}_{21}$. The general method of analyzing systems is to first determine all the forces on each object in the system, then use $\vec{a} = \vec{F}_{Net}/m$ on each object to find the motion of that particular object. The "physics" of the interactions between particles is described by the *force* that one particle exerts on another, which can usually be expressed in a simple way. As objects move, the forces change in time, and the resulting motion can be complicated. The benefit of using Newton's approach is that the equations for the accelerations $\vec{a} \equiv d^2\vec{r}/dt^2$ of objects are much simplier than the equations for the position functions $\vec{r}(t)$. We are lucky that nature turned out to be this simple (at least for the realm of classical mechanics).

The "force-acceleration" approach is helpful in understanding interactions in "classical" mechanics and electromagnetism. However, there are other approaches for analyzing interacting particles. In the next 3 weeks, we consider another approach which deals with the energy and momentum of objects. We will discuss how energy and momentum are related to an objects mass and velocity, and at the same time how the laws of physics can be expressed in terms of energy and momentum. The "force-acceleration" and the "energy-momentum" formalisms contain the same physics. Sometimes one approach is better than the other in analyzing a particular system.

An Example to motivate Mechanical Energy

Consider dropping an object of mass m from rest from a height h above the earth's surface. Let h be small compared to the earth's radius so we can approximate the gravitational force on the object as constant, $m\vec{g}$. The object will fall straight down to the ground. Let y be the objects height above the surface (+ being up) and v be the objects speed. As the object falls, it's speed increases and it's height decreases. That is, y decreases while v increases. Question: is there any combination of the quantities y and v that remain constant?

We can answer this question, since we know how speed v will changes with height y. If the gravitational force doesn't change, the objects acceleration is constant, and is equal to $-g$. The minus sign is because we are choosing the up direction and positive. Let y_i and v_i be the height and speed at time t_i, and let y_f and v_f be the height and speed at time t_f. See Figure 1. From the formula for constant acceleration:

$$v_f^2 = v_i^2 + 2(-g)(y_f - y_i) \tag{1}$$

This equation can be written as:

$$\frac{v_f^2}{2} + gy_f = \frac{v_i^2}{2} + gy_i \tag{2}$$

Wow! This is a very nice result. Since t_i and t_f could be any two times, the quantity $v^2/2 + gy$ does not change as the object falls to the earth! As the object falls, y decreases and v increases, but the combination $v^2/2 + gy$ does not change. As we shall see, it is convenient to multiply this expression by the mass of the object m. Since m also does not change during the fall, we have

$$m\frac{v^2}{2} + mgy = constant \tag{3}$$

while an object falls straight down to earth. In physics when a quantity remains constant in time, we say that the quantity is **conserved**. The quantity that is conserved in this case is the mechanical **energy**. The first term, $mv^2/2$, depends on the particle's motion and is called the kinetic energy. The second term, mgy, depends on the particle's position and is called the potential energy. In these terms, we can say that the sum of the object's kinetic energy plus its potential energy remains constant during the fall.

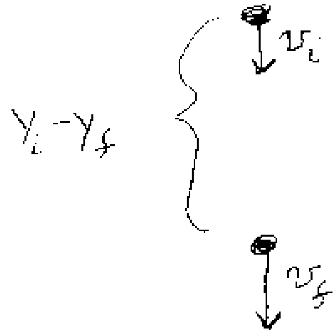
Can the energy considerations for this special case of an object falling straight down be generalized to other types of motion? Yes it can. The situation is a little more complicated in two and three dimensions since **the direction of the net force is not necessarily in the same direction as the objects motion ($\vec{v}$)**. Let's first try another simple case: a block sliding *without friction* down an inclined plane. Let the plane's surface make an angle of θ with the horizontal, and let the block slide down the plane a distance d as shown in Figure 2. The net force down the plane is the component of gravity parallel to the plane: $mgsin\theta$. So the block's acceleration down the plane is $a = (mgsin\theta)/m = gsin\theta$. If the initial speed is v_i and the final speed v_f, we have

$$v_f^2 = v_i^2 + 2ad \tag{4}$$

$$v_f^2 = v_i^2 + 2g(sin\theta)\,d \tag{5}$$

since the acceleration is constant. However, we note from Figure 2 that $dsin\theta$ is just $y_i - y_f$. With this substitution we have

Figure 1



$$v_f^2 = v_i^2 + 2g(Y_i - Y_f)$$

$$v_f^2 = v_i^2 + 2gY_i - 2gY_f$$

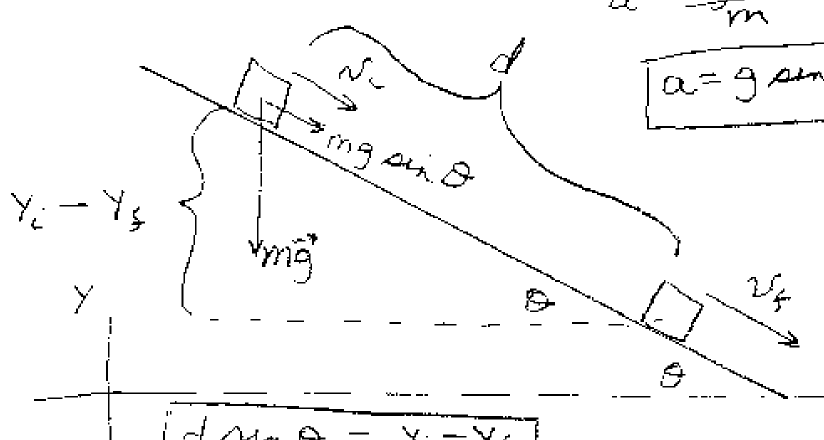
$$v_f^2 + 2gY_f = v_i^2 + 2gY_i$$

$$\frac{mv_f^2}{2} + mgY_f = \frac{mv_i^2}{2} + mgY_i$$

Figure 2

$$a = \frac{mg\sin\theta}{m}$$

$$\boxed{a = g\sin\theta}$$



$$\boxed{d\sin\theta = Y_i - Y_f}$$

$$v_f^2 = v_i^2 + 2a\,d = v_i^2 + 2g\sin\theta\, d$$

$$\boxed{v_f^2 = v_i^2 + 2g(Y_i - Y_f)}$$

$$v_f^2 = v_i^2 + 2g(y_i - y_f) \tag{6}$$
$$v_f^2 + 2gy_f = v_i^2 + 2gy_i \tag{7}$$

Wow, this is the same equation we had before. Multiplying both sides by m and dividing by 2, we obtain

$$\frac{mv_f^2}{2} + mgy_f = \frac{mv_i^2}{2} + mgy_i \tag{8}$$

Thus, as the block slides without friction down the plane, the quantity $(mv^2)/2+mgy$ remains constant.

Is this a general result for gravity near the surface of the earth for any kind of motion? A generalization of this analysis is facilitated by using a mathematical operation with vectors: the "dot" (or "scalar") product. Let's summarize this vector operation, then see how it helps describe the physics.

Vector Scalar Product

The vector scalar product is an operation between two vectors that produces a scalar. Let $\vec{A}$ and $\vec{B}$ be two vectors. We denote the scalar product as $\vec{A} \cdot \vec{B}$. There are a number of ways to derive a scalar quantity from two vectors. One can use $|\vec{A}|$ and $|\vec{B}|$. However, if we define the scalar product as the product of the magnitudes, then the angle does not play a role. If we call θ the angle between the vectors, then we could use cos(θ) or sin(θ) in our definition. If we want $\vec{A} \cdot \vec{B}$ to equal $\vec{B} \cdot \vec{A}$ then we need a trig function that is symmetric in θ. Since cos(-θ) equals cos(θ) it is the best choice. The scalar product is defined as

$$\vec{A} \cdot \vec{B} \equiv |\vec{A}||\vec{B}|cos(\theta) \tag{9}$$

where θ is the angle between the two vectors.

The scalar product can be negative, positive, or zero. It is the magnitude of $\vec{A}$ times the component of $\vec{B}$ in the direction of $\vec{A}$. If the vectors are perpendicular, then the scalar product is zero. The scalar products between the unit vectors are:

$$\hat{\imath} \cdot \hat{\jmath} = \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = 0 \tag{10}$$

and

$$\hat{\imath} \cdot \hat{\imath} = \hat{\jmath} \cdot \hat{\jmath} = \hat{k} \cdot \hat{k} = 1 \tag{11}$$

If the vectors are expressed in terms of the unit vectors (i.e. their components), $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, the distributive property of the scalar product gives

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{12}$$

Work-Energy Theorum

Newton's second law is a vector equation, it relates the acceleration of an object to the net force it experiences, $\vec{F}_{net} = m\vec{a}$. It is also interesting to consider how a scalar dynamical quantity changes in time and/or position. A useful quantity to consider is the speed of an object squared, $v^2 = \vec{v} \cdot \vec{v}$. The time rate of change of v^2 is:

$$\begin{aligned} \frac{dv^2}{dt} &= \frac{d(\vec{v} \cdot \vec{v})}{dt} \\ &= \frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} \\ &= \vec{a} \cdot \vec{v} + \vec{v} \cdot \vec{a} \\ \frac{dv^2}{dt} &= 2\vec{v} \cdot \vec{a} \end{aligned}$$

The result above is strictly a mathematical formula. The last line states that the increase in the speed squared is proportional to the component of the acceleration in the direction of the velocity. This makes sense. If there is no component of $\vec{a}$ in the direction of $\vec{v}$ then the objects speed does not increase. For example in uniform circular motion, $\vec{v}$ and $\vec{a}$ are perpendicular and the speed does not change.

Now comes the physics. Newton's second law relates the acceleration to the *Net Force*, $\vec{a} = \vec{F}_{net}/m$:

$$\begin{aligned} \frac{dv^2}{dt} &= 2\vec{v} \cdot \vec{a} \\ &= 2\vec{v} \cdot \frac{\vec{F}_{net}}{m} \end{aligned}$$

Multiplying both sides by $m/2$ and rearrainging terms gives

$$\vec{F}_{net} \cdot \vec{v} = \frac{d(mv^2/2)}{dt} \tag{13}$$

For an infinitesmal change in time, Δt, we have

$$\vec{F}_{net} \cdot (\vec{v}\Delta t) = \Delta(\frac{mv^2}{2}) \tag{14}$$

However, $\vec{v}\Delta t$ is the displacement of the object in the time Δt, which we will call $\Delta\vec{r} = \vec{v}\Delta t$. Substituting into the equation gives:

$$\vec{F}_{net} \cdot \Delta\vec{r} = \Delta(\frac{mv^2}{2}) \tag{15}$$

This is a very nice equation. It says that the (component of the force in the direction of the motion) times the displacement equals the change in the quantity $mv^2/2$. The quantity on the right side, $mv^2/2$, and the quantity on the left side, $\vec{F}_{net} \cdot \Delta\vec{r}$, are special and have special names. $mv^2/2$ is called the kinetic energy of the object. $\vec{F}_{net} \cdot \Delta\vec{r}$ is called the **Net Work**. Both are scalar quantities, and the equation is a result of Newton's second law of motion. The above equation is the infinitesmal version of the work-energy theorum.

One can integrate the above equation along the path that an object moves. This involves subdividing the path into a large number N of segments. If N is large enough, the segments are small enough such that $\Delta\vec{r}$ lies along the path. Adding up the results of the above equation for each segment gives:

$$\int \vec{F}_{net} \cdot d\vec{r} = \int d(\frac{mv^2}{2}) \tag{16}$$

If the initial position is $\vec{r}_i$ and the final position is $\vec{r}_f$, we have

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r} = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} \tag{17}$$

This equation is called the **work-energy theorum**. It is true for any path the particle takes and derives from Newton's second law. Note that there is no explicit time variable in the equation. The work-energy theorum does not directly give information regarding the time it takes for the object to move from $\vec{r}_i$ to $\vec{r}_f$, nor does it give information about the direction of the object. The equation relates force and the distance through which the net force acts on the object to the change in the objects (speed) squared. It is a scalar equation which contains the physics of the vector equation $\vec{F}_{net} = m\vec{a}$. Since scalar quantities do not have any direction, it is often easier to analyze scalar quantities. From the work-energy theorum *we discover* two important scalar quantities, $mv^2/2$ and $\vec{F}_{net} \cdot \Delta\vec{r}$, and their relationship to each other.

One is called the kinetic energy and the other the **Net Work**. Many situations for which this equation is applicable will be discussed in lecture.

Work done by a particular Force

Although the work-energy theorum applies to the Net Work done on an object, it is often useful to consider the *work done by a particular force*. The word "work" by itself is vague. To talk about work, one needs to specify two things:

1. The **force** (or net force) for the work you are calculating.

2. The **path** that the particle is traveling.

The work done by a particular force $\vec{F}_1$ for a particle moving along a path that starts at location a and ends at location b is written as:

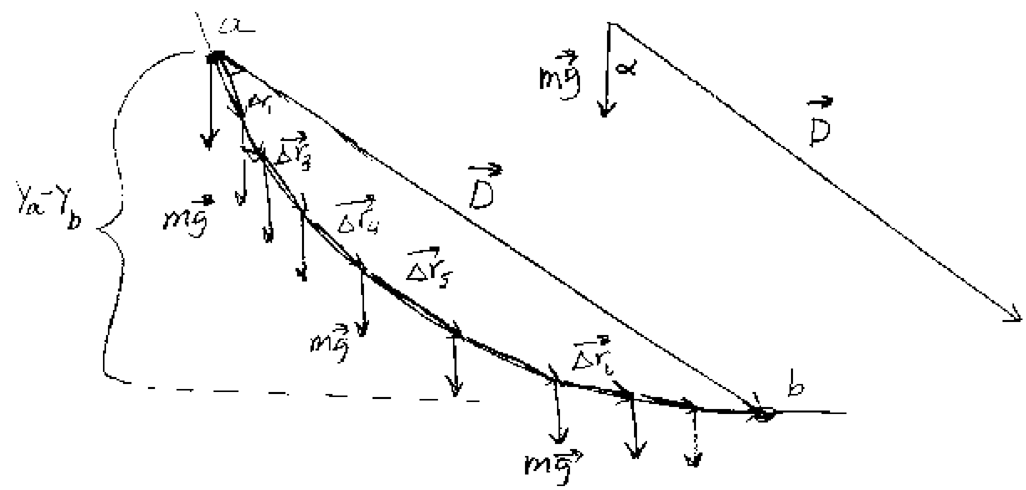
$$W_{a \to b} \equiv \int_a^b \vec{F}_1 \cdot d\vec{r} \tag{18}$$

The integral on the right side of this equation is a "line integral". Generally, line integrals can be complicated. However, in this course we will only consider line integrals that are "easy" to evaluate. We will limit ourselves to simple situations: a) Forces that are constant, and b) for the case of forces that change with position: paths that are in the same direction as the force, and paths that are perpendicular to the direction of the force. In the later case, we will show that the work is zero.

Basically, work is the product of the (force in the direction of the motion) times (the distance the force acts). For a constant force F acting through a distance d, which is in the *same direction* as the force, the work is simply $W = Fd$. The units of work are (force)(distance): Newton-meter, foot-pound, dyne-cm. The work done by a constant force $\vec{F}$ acting through a straight line displacement $\vec{d}$ is just $W = \vec{F} \cdot \vec{d}$. Remember that **Work is a scalar quantity**.

For some forces, the work done only depends on the initial and final locations, and not on the path taken between these two locations. We will demonstrate this by an important example: the work done by the gravitational force near the earths surface. Here we will assume that the force is constant and equal to $m\vec{g}$. Consider the curved path shown in Figure 3. The method of carrying out the integral is to divide up the curved path into small straight segments, which we label as $\Delta\vec{r}_i$. The *work done by the gravitational force from location a to location b along the path shown* is given by

Figure 3



$$Y_a - Y_b = |\vec{D}| \cos\alpha$$

$$W_{a\to b} = m\vec{g}\cdot\Delta\vec{r}_1 + m\vec{g}\cdot\Delta\vec{r}_2 + \cdots$$

$$= \sum_{i=1} m\vec{g}\cdot\Delta\vec{r}_i$$

$$= m\vec{g}\cdot\left(\sum_{i=1}\Delta\vec{r}_i\right) = m\vec{g}\cdot\vec{D}$$

$$= mg|\vec{D}|\cos\alpha$$

$W_{a\to b} = mg(Y_a - Y_b)$ *for any path*

$$W_{a\to b} = \lim_{|\Delta\vec{r}_i|\to 0} \sum_i m\vec{g}\cdot\Delta\vec{r}_i \tag{19}$$

Since the force $m\vec{g}$ is the same for each segment, it is a constant and can be removed from the sum:

$$W_{a\to b} = m\vec{g}\cdot \lim_{|\Delta\vec{r}_i|\to 0} \sum_i \Delta\vec{r}_i \tag{20}$$

Using the "tail-to-tip" method of adding vectors, the sum on the right side is easy. As seen in the figure, adding the $\Delta\vec{r}_i$ tail to tip just gives the vector from location a to location b, which is $\vec{D}$ in the figure.

$$W_{a\to b} = m\vec{g}\cdot\vec{D} \tag{21}$$

We would get this result for **any path** taken from a to b. From Figure 3 we can see that $m\vec{g}\cdot\vec{D}$ is just $mg|\vec{D}|cos\alpha$. However, $|\vec{D}|cos\alpha = y_a - y_b$. So the work done by the graviational force from $a \longrightarrow b$ is just

$$W_{a\to b} = mg(y_a - y_b) \tag{22}$$

We will obtain this same result **for any path the particle moves on from** $a \to b$. We say that the work done by the gravitational force (near the surface) is "path independent", and only depends on the initial and final positions of the particle. Along the path, the "component of the gravitational force" in the direction of the motion can change. However, when added up for the whole path, $W_{a\to b}$ is simply $mg(y_a - y_b)$. Forces that have this "path-independent" property are called **conservative forces**. Conservative forces lead to conserved quantities, which we discuss next.

Potential Energy and Conservation of Mechanical Energy

Often one can identify different forces that act on an object as it moves along its path, and the net force is the sum of these forces: $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + ...$ For example the forces $\vec{F}_j$ can be the weight or gravitational force, the force a surface exerts on an object, an electric or magnetic force, etc. The net work on an object will be the sum of the work done by the different forces acting on the object:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net}\cdot d\vec{r} = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_1\cdot d\vec{r} + \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_2\cdot d\vec{r} + ... \tag{23}$$

or letting W_j represent the work done by the force j:

$$W_{net} = W_1 + W_2 + ... \tag{24}$$

Note this equation motivates us to examine the work done by *one force* as it acts along a particular path from the position $\vec{r}_i$ to $\vec{r}_f$. We first consider the work done by particular forces, then we will discover an energy conservation principle.

Work done by forces that only change direction

Forces that always act perpendicular to the object's velocity will do no work on the object. Since there is no component of force in the direction of the motion, these forces can only change the direction but not increase the speed (kinetic energy) of the object. The tension in the rope of a simple pendulum is an example. If the rope is fixed at one end and doesn't stretch, the tension is always perpendicular to the motion of the swinging ball. If an object slides along a frictionless surface that does not move, the force that the surface exerts, (normal force) is always perpendicular to the motion. If there is friction, the force the surface exerts on an object is often "broken up" into a part normal to the surface and one tangential to the surface. The part of the force "normal" to the surface will not do any work on the object. The tangential part (friction) will do work on the object. Note: if the surface moves, the normal force can do work.

Work done by a constant force

We will consider a particular constant force, the weight of an object near the earth's surface. Our results will apply generally to any force that is constant in space and time. Also, in our discussion we will consider "up" as the "+y" direction. With this notation, the force of gravity is $\vec{W}_g = -mg\hat{j}$. Consider any arbitrary path from an initial position $\vec{r}_i = x_i\hat{i} + y_i\hat{j}$ to a final position $\vec{r}_f = x_f\hat{i} + y_f\hat{j}$. To calculate the work done by $\vec{W}$ from the initial to the final position, we divide up the path into a large number N small segments. We label one segment as $\Delta\vec{r}$. In terms of the unit vectors, we can write $\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$. The work, ΔW_g, done by the force of gravity for this segment is

$$\Delta W_g = (-mg\hat{j}) \cdot (\Delta x\hat{i} + \Delta y\hat{j}) \tag{25}$$

which is

$$\Delta W_g = -mg\Delta y \tag{26}$$

The work done by the force of gravity for this segment does not depend on Δx. This makes sense, since the force acts only in the "y" direction. If all the work done by the gravitational force for all the segments of the path are added up, the result will be

$$\begin{aligned} W_g &= -mg\sum \Delta y \\ &= -mg(y_f - y_i) \\ W_g &= mgy_i - mgy_f \end{aligned}$$

This result is true for any path the object takes. That is, the work done by a constant gravitational force depends only on the initial and final heights of the path. If the work done by a force depends only on the initial and final positions of the path, and not the path itself, the force is called a **conservative force**. The force $\vec{F}_g = -mg\hat{j}$ is a conservative force. We see that the work done by the gravitational force $\vec{F}_g$ equals the difference in the function $U_g(\vec{r}) = mgy$ of the initial and final positions:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_g \cdot d\vec{r} = U_g(\vec{r}_i) - U_g(\vec{r}_f) \qquad (27)$$

The function $U_g(\vec{r})$ is called the **potential energy function** for the constant gravitational force (i.e. gravity near a planet's surface).

In general, whenever the work done by a force depends only on the initial and final positions, and is the same for any path between these starting and ending positions, the work can be written as the difference between the values of some function evaluated at the initial, i, and final, f, positions:

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = U(\vec{r}_i) - U(\vec{r}_f) \qquad (28)$$

This equation is the defining characteristic of conservative forces: the work done by a conservative force from the position $\vec{r}_i$ to the position $\vec{r}_f$ equals the difference in a function $U(\vec{r})$, $U(\vec{r}_i) - U(\vec{r}_f)$. The function $U(\vec{r})$ is called the **potential energy function**, and will depend on the force $\vec{F}$. Every force is not necessarily a conservative one. For example, the contact forces of friction, tension, and air friction are not conservative. The work done by these forces are **not equal** to the difference in a potential energy function.

Note that the potential energy function is not unique. An arbitrary constant can be added to $U(\vec{r})$ and the difference $U(\vec{r}_i) - U(\vec{r}_f)$ is unchanged. That is, if $U(\vec{r})$ is the potential energy function for some force, then $U'(\vec{r}) = U(\vec{r}) + C$ is also a valid potential energy function:

$$\begin{aligned}\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_g \cdot d\vec{r} &= U_g(\vec{r}_i) - U_g(\vec{r}_f) \\ &= (U_g(\vec{r}_i) + C) - (U_g(\vec{r}_f) + C) \\ &= U'_g(\vec{r}_i) - U'_g(\vec{r}_f)\end{aligned}$$

The arbitrary constant C is chosen so that the potential energy is zero at some reference point. For the case of the constant gravitational force, the reference point of zero potential energy is usually where one chooses $y = 0$, i.e. $U_g(\vec{r}) = mgy$. To conclude this section, we mention the potential energy functions for some of the conservative forces that you will encounter during your first year of physics.

Linear restoring force (ideal spring)

For an ideal spring, the force that the spring exerts on an object is proportional to the displacement from equilibrium. If the spring acts along the x-axis and $x = 0$ is the equilibrium position, then the force the spring exerts if the end is displaced a distance x from equilibrium is approximately

$$F_x = -kx \tag{29}$$

The minus sign sigifies that the force is a restoring force, i.e. the force is in the opposite direction as the displacement. If $x > 0$, then the force is in the negative direction towards $x = 0$. If $x < 0$, then the force is in the positive direction towards $x = 0$. The constant k is called the spring constant. The work done by this force from x_i to x_f along the x-axis is

$$\begin{aligned}W_{spring} &= \int_{x_i}^{x_f} -kx dx \\ &= \frac{k}{2}x_i^2 - \frac{k}{2}x_f^2\end{aligned}$$

Thus, the potential energy function for the ideal spring is $U_{spring} = kx^2/2 + C$. The constant C is usually chosen to be zero. In this case the potential energy of the spring is zero when the end is at $x = 0$, i.e. the spring is at its equilibrium position.

$$U_{spring} = \frac{k}{2}x^2 \tag{30}$$

Universal gravitational force

Newton's law of universal gravitation describes the force between two "point" objects: $\vec{F}_{12} = -(Gm_1m_2/r^2)\hat{r}_{12}$ where m_1 and m_2 are the masses of object 1 and 2, r is the distance between the particles, $\vec{F}_{12}$ is the force *on object two* due to object one, $\hat{r}_{12}$ is a unit vector from object one to object two. G is a constant equal to $6.67 \times 10^{-11} NM^2/kg^2$. The minus sign means that the force is always attractive, since m_1 and m_2 are positive.

The graviational force is a "central" force, it's direction is along the line connecting the two objects. This property makes the force conservative. Work is only done by this force when there is a change in r The force does no work if the path is circular, i.e. constant r. The work done by the universal gravity force for paths starting at a separation distance of r_i to a separation distance of r_f is

$$\begin{aligned} W_{U.G.} &= \int_{r_i}^{r_f} -\frac{Gm_1m_2}{r^2} dr \\ &= \frac{Gm_1m_2}{r}\Big|_{r_i}^{r_f} \\ &= \frac{Gm_1m_2}{r_f} - \frac{Gm_1m_2}{r_i} \\ &= -\frac{Gm_1m_2}{r_i} - (-\frac{Gm_1m_2}{r_f}) \end{aligned}$$

Thus, the gravitational potential energy is $U_{U.G.} = -Gm_1m_2/r + C$. The constant C is usually taken to be zero, which sets the potential energy to zero at $r = \infty$.

$$U_{U.G.} = -\frac{Gm_1m_2}{r} \tag{31}$$

Electrostatic interaction

Coulomb's law describes the electrostatic force between two point objects that have charge: $\vec{F}_{12} = (kq_1q_2/r^2)\hat{r}_{12}$ where q_1 and q_2 are the charges of object 1 and 2, r is the distance between the particles, $\vec{F}_{12}$ is the force *on object two* due to object one, $\hat{r}_{12}$ is a unit vector from object one to object two. The constant k equals $9 \times 10^9 NM^2/C^2$. If the objects have the same sign of charge, the product q_1q_2 is positive and the force is repulsive. If the objects have opposite charge, the product q_1q_2 is negative and the force is attractive. The force has the same form as the universal gravitational force.

Charge is the source of the force instead of mass. Similar to the gravitation case, the work done by the electrostatic force for paths starting at a separation distance of r_i to a separation distance of r_f is

$$\begin{aligned} W_{electrostatic} &= \int_{r_i}^{r_f} \frac{kq_1q_2}{r^2} dr \\ &= -\frac{kq_1q_2}{r}\Big|_{r_i}^{r_f} \\ &= -\frac{kq_1q_2}{r_f} - (-\frac{kq_1q_2}{r_i}) \\ &= \frac{kq_1q_2}{r_i} - \frac{kq_1q_2}{r_f} \end{aligned}$$

Thus, the electrostatic potential energy is $U_{electrostatic} = kq_1q_2/r + C$. The constant C is usually taken to be zero, which sets the potential energy to zero at $r = \infty$.

$$U_{electrostatic} = \frac{kq_1q_2}{r} \tag{32}$$

Conservation of Mechanical Energy

The reason for using the word "conservative" to describe these forces is the following. Suppose that the only forces that do work on an object are ones that are conservative, that is, the work done by these forces is equal to the difference in a potential energy function. Consider the case in which the only forces that do work on a particle are two conservative forces. From the work-energy theorum we have:

$$\begin{aligned} \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r} &= \frac{m}{2}v_f^2 - \frac{m}{2}v_i^2 \\ \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_1 \cdot d\vec{r} + \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_2 \cdot d\vec{r} &= \frac{m}{2}v_f^2 - \frac{m}{2}v_i^2 \\ (U_1(\vec{r}_i) - U_1(\vec{r}_f)) + (U_2(\vec{r}_i) - U_2(\vec{r}_f)) &= \frac{m}{2}v_f^2 - \frac{m}{2}v_i^2 \end{aligned}$$

Rearrainging terms we have:

$$U_1(\vec{r}_i) + U_2(\vec{r}_i) + \frac{m}{2}v_i^2 = U_1(\vec{r}_f) + U_2(\vec{r}_f) + \frac{m}{2}v_f^2 \tag{33}$$

The left side of the equation contains only quantities at position $\vec{r}_i$, and the right side only quantities at position $\vec{r}_f$. Since $\vec{r}_f$ is arbitrary, the quantity $U_1(\vec{r}) + U_2(\vec{r}) + mv^2/2$ is a **constant of the motion, it is a conserved quantity!** The sum of the **potential energy** plus **kinetic energy**, which we call the **total mechanical energy**, is conserved. This result will hold for any number of forces, as long as the only work done on the system is by conservative forces. WOW!!

If non-conservative forces act on the object, such as frictional forces, the total mechanical energy as defined above is not conserved. However, frictional forces are really electro-magnetic forces at the microscopic level. The electro-magnetic interaction is conservative if the energy of the electromagnetic field is included. At present, we believe there are only three fundamental interactions in nature: gravitational, electro-magnetic-weak, and strong, and they all are conservative. Frictional forces are non-conservative because of the limited description used in analyzing the system of particles. Since all the fundamental forces of nature (gravity, electro- magnetic, and strong) are conservative, at the microscopic level total energy is conserved. We refer to the energy of the all the atoms and molecules of a material as it's **internal energy**.

Momentum and Conservation of Momentum

An example to motivate the concept of momentum

There is a physics professor holding on to a cart. The physics professor, mass m_1, is wearing frictionless roller skates, and the cart (mass m_2) has frictionless wheels. The professor pushes the cart with a constant force F, he goes off to the left (negative direction) and the cart goes off to the right (positive direction). How is the professor's final speed, $|v_1|$, related to the final speed of the cart, $|v_2|$? From **Newton's Third Law**, the force the cart feels due to the professor, F_{CP}, is equal in magnitude and opposite in direction to the force the professor feels due to the cart, F_{PC}:

$$-F_{PC} = F_{CP} \tag{34}$$

The time that the force acts, t, is the same for both objects. That is, the time the force acts on the professor is the same as the time for force acts on the cart:

$$-F_{PC}t = F_{CP}t \tag{35}$$

Using Newton's second law, $\vec{F}_{net} = m\vec{a}$, we have

$$-m_1 a_1 t = m_2 a_2 t \tag{36}$$

Since the objects both stated from rest $v = at$ for each object, gives

$$-m_1 v_1 = m_2 v_2 \tag{37}$$

Wow, what a simple result. The magnitude of mv for the professor to the left equals the magnitude of mv for the cart to the right. This result seems to work for any force and any time t. The quantity mass times velocity appears to be quite special, and deserves a special name. We call the product of mass times velocity the momentum of the particle, $\vec{p}$. For example, the **momentum** of particle 1 is

$$\vec{p}_1 \equiv m_1 \vec{v}_1 \tag{38}$$

Note that momentum is a vector!

Some properties of the momentum of a particle

Newton's law of motion for a particle can be expressed in terms of the momentum of the particle:

$$\begin{aligned} \vec{F}_{net} &= m\vec{a} \\ &= m\frac{d\vec{v}}{dt} \\ &= \frac{d(m\vec{v})}{dt} \\ \vec{F}_{net} &= \frac{d\vec{p}}{dt} \end{aligned}$$

If one multiplies both sides by dt and integrate from time t_1 to time t_2, we have

$$\int_{t_1}^{t_2} \vec{F}_{net} dt = \vec{p}_2 - \vec{p}_1 \tag{39}$$

This equation is called the **"impulse-momentum" theorum**. It is similar to the work-energy theorum. Loosely speaking: *force times time equals the change in momentum* of a particle, and *force times distance equals the change in kinetic energy* of a particle.

Total Momentum of a system of particles

Often in physics we are interested in how particles interact with each other. The number of interacting particles can be two or more, and we refer to the collection of particles as a system of particles. An important quantity to consider for a system of particles is the **total momentum of the particles**, $\vec{P}_{tot}$. The total momentum of the system of particles is just the vector sum of the momenta of the individual particles:

$$\vec{P}_{tot} \equiv \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + ... \tag{40}$$

where the subscripts 1, 2, etc. refer to particle number 1, 2, etc. How does the total momentum of the system of particles change in time? Consider the case of two particles:

$$\begin{aligned} \frac{d\vec{P}_{tot}}{dt} &= \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} \\ &= \vec{F}_{1net} + \vec{F}_{2net} \end{aligned}$$

We can separate the net force on each particle into a part due to the particles within the system and a part due to influences outside the system. In the case of two particles, the net force on particle one equals the force on particle one due to particle two, $\vec{F}_{12}$, and the force due to objects outside the system, which we call external forces: $\vec{F}_{1ext}$. The same is true for object 2:

$$\begin{aligned} \vec{F}_{1net} &= \vec{F}_{12} + \vec{F}_{1ext} \\ \vec{F}_{2net} &= \vec{F}_{21} + \vec{F}_{2ext} \end{aligned}$$

Substituting the net force equations into the change of total momentum equation above gives:

$$\begin{aligned} \frac{d\vec{P}_{tot}}{dt} &= \vec{F}_{12} + \vec{F}_{1ext} + \\ & \quad \vec{F}_{21} + \vec{F}_{2ext} \end{aligned}$$

Two terms on the right side of the equation ($\vec{F}_{12}$ and $\vec{F}_{21}$) cancel each other. From Newton's third law, $\vec{F}_{12} = -\vec{F}_{12}$. The force particle *one* feels due to particle *two* is equal in magnitude and opposite in direction to the force that particle *two* feels from particle *one*. Although $\vec{F}_{12}$ and $\vec{F}_{21}$ act on different particles, when all the forces of the whole system are added up they cancel each other out. Thus we have

$$\frac{d\vec{P}_{tot}}{dt} = \vec{F}_{1ext} + \vec{F}_{2ext} \tag{41}$$

If the system contains more than two particles, the result will be similar. In general, we have:

$$\frac{d\vec{P}_{tot}}{dt} = \vec{F}_{1ext} + \vec{F}_{2ext} + ... \tag{42}$$

The time rate of change of the total momentum of a system of particles equals the sum of the external forces. **If there are no external forces acting on the system**:

$$\frac{d\vec{P}_{tot}}{dt} = 0 \tag{43}$$

That is, the total momentum of the system does not change in time, it is conserved.

$$\vec{P}_{tot} = constant \tag{44}$$

This is a very nice result, and it is straight forward to see why it is true in the two particle case. The key physics is Newton's third law. If there are no external forces, the net force on object 1 is caused only by object 2 (and visa-versa). Whatever force object 1 feels, object 2 will feel the opposite force. Since the time of interaction, Δt is the same for both particles, $\vec{F}_{12}\Delta t = -\vec{F}_{21}\Delta t$. Force times time is the change in momentum, thus the vector change in the momentum of particle one will be opposite to the vector change in momentum of particle 2: $\Delta\vec{p}_1 = -\Delta\vec{p}_2$. Object 1's gain (loss) of momentum equals object 2's loss (gain). The net change in the total momenta of the two objects is zero. This same argument is true if there are more then two objects.

In lecture we will discuss several cases where there are no external forces: collisions, explosions, etc. When no external forces act in a collision, the total momentum (**vector sum** of the momenta of the particles) of the system is conserved. If kinetic energy is also conserved, we call the collision elastic. It is remarkable that one doesn't need to know the details of the collision. As long as there are no external forces, it doesn't matter what kind of forces are involved: the total momentum remains constant before, during and after the collision.

In the derivation above, **the conservation of total momentum** comes from Newton's third law, which **is a result of a symmetry in nature**. The interaction between object 1 and object 2 is symmetric, they each must experience the same force.

Is this a general result, that symmetries in nature lead to conserved quantities? We find that in many cases this is true. Rotational symmetry leads to conservation of angular momentum, and momentum and energy conservation are a result of space and time symmetry. This is a grand idea, and helps in describing the physics of subatomic particles. We do experiments to identify conserved quantities, then develop mathematical descriptions that have the corresponding symmetry properties. The connection between symmetries in nature and conserved quantities is one of the more "beautiful" principles of physics.

Center of Mass of a system of particles

From the total momentum of a system of particles, we can define another special quantity for a system of particles: the "center-of-mass" velocity. The center-of-mass velocity, $\vec{V}_{cm}$, is defined as the total momentum divided by the total mass of the system:

$$\vec{V}_{cm} \equiv \frac{\vec{P}_{tot}}{M_{tot}} \tag{45}$$

where the total mass of the system, M_{tot} is defined as

$$M_{tot} \equiv m_1 + m_2 + ... \tag{46}$$

The center-of-mass velocity is easily expressed in terms of the individual masses and velocities of the particles that make up the system:

$$\vec{V}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + ...}{m_1 + m_2 + ...} \tag{47}$$

Note that the center-of-mass velocity is a vector. From the center-of-mass velocity, it is straight forward to define a center-of-mass position and a center-of-mass acceleration:

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + ...}{m_1 + m_2 + ...} \tag{48}$$

for the center-of-mass acceleration, and

$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + ...}{m_1 + m_2 + ...} \tag{49}$$

for the center-of-mass position. $\vec{R}_{cm}$ is usually called center-of-mass. The connection between these three "kinematical" quantities is the same as with the position, velocity, and acceleration of one particle:

$$\begin{aligned}\vec{V}_{cm} &= \frac{d\vec{R}_{cm}}{dt} \\ \vec{a}_{cm} &= \frac{d\vec{V}_{cm}}{dt}\end{aligned}$$

These center-of-mass quantities have some nice properties. Since $\vec{P}_{tot} = M_{tot}\vec{V}_{cm}$, we have

$$\begin{aligned}M_{tot}\frac{d\vec{V}_{cm}}{dt} &= \frac{d\vec{P}_{tot}}{dt} \\ &= \vec{F}_{ext-net}\end{aligned}$$

where $\vec{F}_{ext-net}$ is the sum of the external forces. This equation can be rewritten as

$$\frac{d\vec{V}_{cm}}{dt} = \frac{\vec{F}_{ext-net}}{M_{tot}} \tag{50}$$

Thus, the acceleration of the center-of-mass equals the net force divided by the total mass. **If there are no external forces**:

$$\begin{aligned}\frac{d\vec{V}_{cm}}{dt} &= 0 \\ \vec{V}_{cm} &= constant\end{aligned}$$

Wow, another nice result. If there are no external forces, the center-of-mass of the system moves at a constant velocity. If it is initially at rest (in an inertial frame) it remains at rest. This result also defines a special reference frame for a system of particles: the reference frame for which the center-of-mass does not move.

If there are no external forces acting on a system of particles, There exists an inertial frame of reference in which the center-of-mass $\vec{R}_{cm}$ remains at rest. We refer to this reference frame as **the center-of-mass reference frame**, or simply center-of-mass frame. Although the individual particles in a system might move in a complicated manner, there is one special position, the center-of-mass, which moves in a simple way. It is often easier to analyze the motion of a system of particles from the center-of-mass frame.

Final comments on Energy and Momentum

The first 4 weeks of the quarter we discussed the physics of the interaction of particles using the "force-motion" approach: find the net force on each object, then $\vec{a} = \vec{F}_{net}/m$. The symmetry of interaction was expressed via Newton's third law. During the last 3 weeks, we expressed these laws using the energy and momentum of the particles. The two approaches are equivalent. Since we believe the fundamental forces are conservative, from the potential energy functions the forces can be determined and visa-versa:

$$F_x = -\frac{\partial U}{\partial x}$$
$$F_y = -\frac{\partial U}{\partial y}$$
$$F_z = -\frac{\partial U}{\partial z}$$

What are the advantages of using the "energy-momentum" approach verses the "force-motion" approach? In terms of solving problems, if one is interested in how an objects speed changes from one point to another then the energy approach is much simplier than solving for the acceleration. However, *problem solving is not the main motivation for studying energy and momentum.*

The laws of physics for "small" systems (atoms, nuclei, subatomic particles) and for relatively fast objects (special relativity) are best described in terms of momentum and energy. Energy and momentum transform from one inertial reference frame to another the same way as displacements in time and space. The interference properties of a particle is related to the particle's momentum ($\hbar/p$). In the Schrödinger equation, which is used to describe quantum systems, the interaction is expressed in terms of the potential energy of the system. Advances in modern physics were guided by the principles of energy and momentum conservation.

Energy plays a key role in life on earth. If an animal needs to use more energy obtaining food than the food supplies, then the animal starves and species can go extinct. Energy is an important commodity in our daily lives: for our bodies, for our homes, and for our personal transportation. Wars have been fought over energy, and our lifestyle will be determined by how successful we are in using the sun's energy.

Energy is perhaps more important than momentum because it is a scalar quantity allowing it to be stored and sold. The importance of energy cannot be understated. Next quarter, a large part of your physics course (Phy132) will be devoted to an understanding of internal energy and energy transfer processes.