

Notes on Introductory Mechanics

These notes are meant to help students understand the basic "physics" behind introductory "Newtonian" mechanics. The ideas are presented in the order in which they are taught in my first year class, and are designed to supplement the text.

The approach we take is somewhat historical. We first consider how forces combine such that a point particle does not move: the statics of forces acting at a point. Then we investigate the relationship between force and motion in a one dimension frictionless world. Next, two and three dimensional motion is discussed. We begin with the motion of a "point particle", and then consider the possibility for objects to rotate.

Combining Forces

In early times the balancing of forces was important for building, and was the first "physics" to be investigated. Later on in the quarter we will analyze mechanical systems precisely. We will discuss the concepts of inertial mass, force and how forces cause motion, ideas that were discovered in the 1600's. For now, we will consider a simple set of experiments with forces.

Simply speaking, a force is a push (or pull). Let's consider forces due to an objects weight, i.e. the gravitational force due to the attraction of the earth. For this discussion, we limit ourselves to objects that are made of pure elements, i.e. copper, iron, etc. We will make the assumption that for a pure homogenous substance the weight is proportional to the volume of the object. That is, if object A is twice as large object B, and both are made of the same substance, then object A has twice the weight as object B. If one object has a weight of 10 units, then an object of twice (or x times) the volume will have a weight of 20 units (or $10x$ units). The weights in our weight set are so labeled.

Combining forces in the same direction

In our experiments we will combine forces on a ring by using pulleys and strings. A string attached to the ring will pass over a pulley. The other end of the string will be attached to a weight. Suppose two weights, each of magnitude 10 units, pulls on the ring to the right (the $+$ direction). How much weight and where should it be placed to balance the two weights of 10 units. Your guess is probably one weight of magnitude 20 units pulling to the left. We will check this result in class. (Your guess was correct). This means that two forces pulling in the same direction add like real numbers. Similarly, we will show that two forces pulling in the opposite direction

subtract. Thus for the case of one dimension, we can assign a (+ or -) to a force to signify the direction (right or left).

Combining forces in different directions

A forces can act in any direction. What role does the direction of a force play when combining forces? Consider the following experiment:

A force of 40 units pulls on the ring to the east and a force of 30 units pulls on the ring to the north. What force on the ring will balance these two forces?

After doing the experiment in class, we will find that the balancing force is a force that pulls on the ring with an amount of 50 units at a direction of 36.9° S of W. Thus the force of 40 units to the east plus the force of 30 units to the north is equivalent to a single force of magnitude 50 units at an angle of 36.869° N of E. In this case 30 plus 40 equals 50. You probably remember from trig that lengths of 30, 40, and 50 form a right triangle (3-4-5 triangle). Thus, if we represent each force by an arrow, whose length is proportional to its magnitude and whose direction is in the direction of the force, the two forces combine by placing the tail of one at the tip of the other. The resultant force is represented by the arrow along the hypotenuse.

In lab, you will carry out a number of experiments that demonstrate that any two forces combine using the "arrow" method described above. The mathematics that best describes how forces combine is the mathematics of *vector addition*. Since a number of physical quantities have the properties of mathematical vectors, we will spend some time in lecture discussing vectors.

Vectors are discussed in many texts and usually are defined (initially) as something with direction and magnitude. Quantities with direction and magnitude are not necessarily vectors. They must also have certain mathematical properties. An addition operation must be defined, and the sum of two vectors must also be a vector. If \vec{A} and \vec{B} represent any two vectors, then it must be true that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Vectors are also defined over a field, which in our class will be the real numbers. To be an "inner-product" vector space, a scalar product between two vectors must be defined with certain properties. We will talk about this later when we discuss energy.

The experimental result pertaining to adding "static" forces is summarized by the following experiment:

Experiment: If two forces, \vec{F}_1 and \vec{F}_2 , act on an object that doesn't move, the

resulting force is the same as if the object were subject to the force \vec{F} where \vec{F} is the vector sum of the two forces: $\vec{F} = \vec{F}_1 + \vec{F}_2$.

This is a wonderful experimental result! Forces didn't have to "add" in such a simple way, but they do. It demonstrates that certain aspects of nature can be understood using geometry (trig). These ideas were developed over 3000 years ago, and helped in the building of magnificent structures. However, it took till the 1600's for scientists to correctly understand motion. Motion involves understanding velocity, acceleration, and force. The effect of forces on an object's motion was miss-understood for thousands of years. We will devote most of the next 8 weeks developing these concepts and analyzing the relevant experiments. We start first with one dimensional motion in a straight line, then extend the ideas to two and three dimensions.

One Dimensional (straight-line) Frictionless Environment

Describing Motion: Position, Time, Velocity

To describe motion of a point particle in one "straight-line" dimension one needs to come up with a measure of two quantities: position (or distance) and time.

To measure **position**, we set up a coordinate system, which in one dimension is just a straight line with a reference point chosen as the origin. Once a direction for positive distance and a unit length have been decided upon, equal lengths can be marked on the line. A particles position is specified by a real number (+ or -) indicating its location on the line. Two systems of units are used in the class, MKS and British. Length is measured in meters in the MKS, and in units of feet in the British system.

To measure **time**, we need an instrument (a clock) that can produce equal time intervals. Whereas we have a physical feeling for equal distances from our hands, arms and eyes, equal time intervals are difficult to get a feeling for. Physiologically a good movie lasting as long as a physics lecture might seem much shorter in time. Therefore we can't rely on our senses to judge equal time intervals. One could use a pendulum or other oscillating device. However, using the motion of a physical system to determine equal time intervals and then using this clock to understand motion itself might bias our description and laws of physics. We might wonder if our method of measuring time is making our equations of physics to complicated. It would be nice to have a clock that does not rely on a physical system. We will discuss such a possibility in the next section. In order not to get caught up in circular arguments and deep philosophy, in this class we will accept that clocks can be made

that give equal time intervals. As discussed in the text, the accepted definition of a second is 9192631770 oscillations of a particular transition in the Cesium atom. Although the atom is a physical system, we obtain a workable definition of time for developing physics theories. The interested student should take our modern physics course, where Einstein's theory of special relativity addresses these problems.

Once we have established distance and time units, we can define a **position function**, $x(t)$. The position function is just the position x for the particle as a function of time, t .

A particles **displacement** between the times t_1 and t_2 is just equal to $x(t_2) - x(t_1)$, where $t_2 > t_1$. If the displacement is positive, the particle has moved to the right (+ direction) from the time t_1 to the time t_2 . If the displacement is negative, the particle has moved to the left (- direction). To talk about displacement, you need to refer to *two times*.

Velocity is a measure of how fast an object is moving. **Average velocity**, $\bar{v}_{t_1 \rightarrow t_2}$, is defined to be the displacement/(time interval):

$$\bar{v}_{t_1 \rightarrow t_2} \equiv \frac{x(t_2) - x(t_1)}{t_2 - t_1} \quad (1)$$

As with displacement, to calculate the average velocity of an object one needs to specify the *two* times, t_1 AND t_2 . Average time is not a particularly good quantity to express the laws of mechanics in a simple way. For understanding the laws of physics in a simple way, the velocity at an instant, instantaneous velocity, is a much better quantity. **Instantaneous velocity** at the time t_1 , $v(t_1)$, is defined to be:

$$v(t_1) \equiv \lim_{t_2 \rightarrow t_1} \frac{x(t_2) - x(t_1)}{t_2 - t_1} \quad (2)$$

This can also be written as:

$$v(t_1) \equiv \lim_{\Delta t \rightarrow 0} \frac{x(t_1 + \Delta t) - x(t_1)}{\Delta t} \quad (3)$$

which you will recognize as the first derivative of $x(t)$ evaluated at the time t_1 : $v(t_1) \equiv (dx/dt)|_{t=t_1}$. A nice thing about instantaneous velocity is that it is defined at a single time, t_1 . For this reason it is a better quantity to use than average velocity to try and describe how nature behaves. If the instantaneous velocity is positive (negative) then the particle is moving to the right (left) at that instant.

A particle's **speed** is defined to be the magnitude (or absolute value) of the instantaneous velocity.

Special Case of Constant Velocity

In some special cases, a particles velocity may be constant for a long period of time. If this is so, then the position function takes on a simple form. Let v_0 be the velocity of the particle at the time when we start our clocks, $t = 0$. We will often use the subscript 0 to label the value of a variable at time $t = 0$. For a particle moving with a constant velocity, v_0 will be the velocity for all times t . From the definition of velocity: $(x(t) - x(0))/(t - 0) = v_0$. If we define $x(0) \equiv x_0$, we have the simple relationship for $x(t)$:

$$x(t) = x_0 + v_0 t \quad (4)$$

Often one writes $x(t)$ as just x , so the equation becomes $x = x_0 + v_0 t$.

Under what situations will an object move with a constant velocity in a straight line? A person walking in a straight line down the street with a constant speed, or a car driving down a straight length of road with a constant speed are some examples that come to mind. However, there is a very important case related to a fundamental Law of Motion.

Consider a "reference frame" which is a box floating in space far from any other objects. If you were to go inside this box, floating in space, you would feel "at rest". You could not sense that you were moving. You set up a coordinate system to measure position (one dimensional) and time. Suppose a particle had an initial velocity v_0 with respect to your coordinate system. There are no forces on the particle. What would happen to the particle in time? Would it come to rest, or continue to move with the velocity v_0 in a straight line? Newton and Galileo realized that the particle would continue to move with the velocity v_0 . If the particle was at rest ($v_0 = 0$) it would remain at rest. This property of motion is referred to as Newton's first Law of Motion:

If there are no forces acting on an object, an object at rest remains at rest and an object in motion continues in a state of uniform motion

This idea might seem simple to us, but at the time it was proposed it was profound. It was believed that the natural state of an object was at rest, and that objects that were moving came to rest on their own. Newton's first law of motion applies to reference frames that are floating in space. Only in these frames will an object that

is released at rest in "mid air" stay at rest. The name we give to reference frames for which this law (Newton's first law) of motion holds is an **inertial reference frame**.

Imagine two reference frames floating in space. Suppose someone named Bill was in one, and George in the other. Suppose that George observed that Bill was moving in the + direction with a *constant velocity* $+v$. George would feel at rest and say that Bill was moving to the right with a constant velocity. Bill, however, would also feel at rest and say that George is moving to the left with a constant velocity $-v$. Who is correct? Both are. Each of these frames is an inertial reference frame. Bill feels at rest, and so does George. There is something very special about reference frames floating in space with a constant velocity with respect to each other. They are all inertial reference frames and have the following properties:

1. A reference frame moving with a constant velocity with respect to an inertial frame is also an inertial reference frame.
2. In an inertial reference frame, one "feels" at rest.
3. There is no experiment that one can do in an inertial reference frame to determine the velocity of the reference frame.
4. The laws of physics take on the same form in all inertial reference frames.
5. There is no absolute reference frame.

The equivalence of inertial reference frames is a fundamental property of physics, and is the basis of Einstein's theory of special relativity. It is a wonderful property of nature, and one can marvel at its simplicity.

A final note on Newton's first law is that it allows one to define equal time intervals independent of a physical system. Here is how to do it: Set up your "x" axis, pick an origin, and a unit length. Use your unit length to mark on your "x" axis equal distances. Then set an object (with no forces) in motion. It will float along your "x" axis. A time interval occurs each time that it passes a mark. According to Newton's first law the time intervals will be equal.

Describing Motion: Acceleration

Objects don't always move with constant velocity, velocities change. The change in velocity per unit time is called acceleration. The average acceleration, $\bar{a}_{t_1 \rightarrow t_2}$,

between time t_1 and t_2 is defined to be

$$\bar{a}_{t_1 \rightarrow t_2} \equiv \frac{v(t_2) - v(t_1)}{t_2 - t_1} \quad (5)$$

As with average displacement and average velocity, one needs *two times* to calculate the average acceleration. Since two times are needed, the average acceleration is not a good quantity to describe the laws of physics. A more useful quantity is the instantaneous acceleration, which is defined in the same way as instantaneous velocity. One takes the limit of the average acceleration as t_2 approaches t_1 :

$$a(t_1) \equiv \lim_{t_2 \rightarrow t_1} \frac{v(t_2) - v(t_1)}{t_2 - t_1} \quad (6)$$

Instantaneous acceleration is also written as:

$$a(t_1) \equiv \lim_{\delta t \rightarrow 0} \frac{v(t_1 + \delta t) - v(t_1)}{\delta t} \quad (7)$$

The limit on the right side is just the first derivative of the velocity evaluated at the time t_1 . So $a(t_1) \equiv dv/dt|_{t=t_1}$. Often the t_1 is replaced by t , and one writes $a \equiv dv/dt$. A nice property about instantaneous acceleration is that it is determined at *one time*. The acceleration is just the second derivative of position with respect to time: $a = d^2x/dt^2$.

One could also consider more derivatives such as da/dt , d^2a/dt^2 , etc... to describe motion. We need to rely on experiments to determine the simplest way to relate interactions (forces) and motion.

If the position function, $x(t)$, is known, it is easy to find $v(t)$ and $a(t)$ by differentiation. If one knows $a(t)$, one needs to integrate with respect to t to find $v(t)$. To find $x(t)$, one needs to integrate $v(t)$ with respect to t . A simple case that is discussed in many texts is the special case of motion with constant acceleration. Suppose the acceleration of a particle is constant, label it a_0 . Then, $dv/dt = a_0$. Multiplying both sides by dt and integrating we have $\int_0^t dv = \int_0^t a_0 dt$, which gives $v(t) - v_0 = a_0 t$ where v_0 is the velocity at $t = 0$, $v(0)$. This is often written as $v = v_0 + a_0 t$, where v means $v(t)$.

To solve for $x(t)$ for the case of constant acceleration requires one more integration. From $dx/dt = v(t) = v_0 + a_0 t$, one can multiply both sides by dt and integrate: $\int_0^t dx = \int_0^t v_0 dt + \int_0^t a_0 t dt$. After integrating, one obtains: $x(t) - x(0) = v_0 t + a_0 t^2/2$. This equation is often written as $x = x_0 + v_0 t + a_0 t^2/2$, where x means $x(t)$.

Summarizing for the *special case of constant acceleration*:

$$v = v_0 + a_0t \quad (8)$$

$$x = x_0 + v_0t + \frac{a_0}{2}t^2 \quad (9)$$

Eliminating t from these two equations gives:

$$v^2 = v_0^2 + 2a_0(x - x_0) \quad (10)$$

In general, motion that has constant (non-zero) acceleration for extended periods of time is fairly rare. Under certain circumstances however, the motion of an object can be *approximately* one of constant acceleration for a period of time. These special cases include situations where the net force is *approximately* constant (e.g. free fall near the earth's surface in the absence of air friction, rolling or sliding down a ramp with no friction, ...). Since the equations for position and velocity are simple for motion of constant acceleration, many texts use examples and problems of this special case to test the student's understanding of these concepts.

One final note: the sign's of $x(t)$, $v(t)$, and $a(t)$ are not related to each other. The acceleration can be in the negative (positive) direction eventhough the velocity is positive (negative), etc. The sign of a is in the direction of the change of v .

Forces, Inertia and Motion

With the mathematics of calculus, which enables us to work with instantaneous rates of change, we can formulate the connection between force and motion. We will start with the simple case of one dimensional motion, which takes place in an inertial reference frame in the absence of friction. Our perception of **force** is that it is a push or a pull. We also have experienced that it is easier to change the motion of a "smaller" object than a "larger" one. We call **inertia** the resistance to a change in motion. **Inertial mass** is a measure of how much inertia an object has. Physics is a mathematical science, and we would like to see if there is a quantitative relationship between force, inertial mass and motion. After doing some experiments, we will try to formulate a theory. This means that we need to determine a method of quantifying (obtaining a number value for) force and inertial mass as well as a mathematical relationship between these quantities.

We should start by doing simple experiments. First we need to agree on a standard for inertial mass. We pick a certain object (standard mass) and call it's inertial mass m_0 (e.g. one kilogram). Let's examine what happens when the standard mass is subject to a **constant force**. How can one apply a constant force to an object in an inertial reference frame (floating in space)? One way is to use a "perfect" spring and

pull on the object such that the spring's extension is constant. I think we can agree that if a spring is stretched by a *fixed amount* the force it exerts will not change. By a "perfect" spring, we mean just that, that the spring does not weaken over time but keeps the same constant force. What motion might result? Most likely not constant velocity, since this is the case if there are no forces. The object will accelerate, but will the acceleration change in time? We need to do the experiment, and will perform a similar one in lecture using the air track. Here is the result from the experiment:

Experiment 1: If an object is subject to a *constant force*, the motion is one of *constant acceleration*.

This is an amazing result! Nature didn't have to be this simple. The acceleration might have changed in time with a constant force, but within the limits of the experiment it doesn't. Experiments show that this result is true for any object subject to any constant force. I should note that this experimental result is only valid within the realm of non-relativistic mechanics. If the velocities are large and/or the measurements very very accurate, relativistic mechanics are needed to understand the data.

The results of experiment 1 allow us to quantify force such that the acceleration is proportional to the force. We can do this because for a fixed force there is a unique acceleration, the acceleration doesn't change in time. To quantify force, we first pick a standard force, F_0 , from a certain spring extension. Then apply this standard force to our standard mass and measure the acceleration a_0 . To determine the amount of another force F , apply F to the standard mass and measure the acceleration, a . Then $F = F_0(a/a_0)$.

We need another experiment determine how to quantify inertial mass and its relationship to acceleration and force. Take two identical objects. Apply the constant force F_0 to one object. The acceleration will be constant, call it a_1 . Now connect the two identical objects together and apply the same force F_0 to both. The acceleration is constant, but how large is it? We will do a similar experiment in lecture. The result from the experiment is:

Experiment 2: If a constant force is applied to two identical objects which are connected to each other, the measured acceleration is it 1/2 the acceleration of one of the objects if it is subject to the same constant force.

This is also an amazing result, and didn't have to be so simple! It shows that if the amount of material is doubled, the acceleration is cut in half for the same constant force. Experiment will also show that x times the amount of material results in an acceleration equal to $1/x$ times a_1 . Inertial mass is proportional to the amount of material for a homogeneous object. Experiment 2 gives us a way to quantify inertial mass. Decide on a reference mass m_0 (e.g. one kilogram). Pick a constant spring extension which will produce a constant force. The constant force applied to the reference mass, causes an acceleration (a_0). Now apply the same constant force to the (unknown) mass being measured. The acceleration will be constant, a . The inertial mass, m , of the unknown is $m = m_0(a_0/a)$.

A similar experiment is the following: **Experiment 2b:** Apply a constant force to object 1, and call the acceleration a_1 . Apply the same force to object 2 and call the acceleration a_2 . Attach the two objects together and apply the same force. Call the acceleration of the two connected objects a_{12} . Experiment will show that $1/a_{12} = 1/a_1 + 1/a_2$ for any two objects and any constant force.

Experiments 1 and 2 allow the quantification of (a single) force and mass such that force equals mass times acceleration: $F = ma$. As far as we know, inertial mass is an intrinsic property of an object and therefore will have its own units. In the MKS system, the unit is the kilogram (Kg). If a is in units of M/s^2 , then force will have units of KgM/s^2 , and is a derived quantity. One KgM/s^2 is called a Newton (N). If a constant force of one Newton is applied to an object whose inertial mass is 1 Kg, the object has a constant acceleration of $1 M/s^2$. For any single force, F , acting on an object of mass m , the acceleration is $a = F/m$. More force produces more acceleration, more mass results in less acceleration.

We need one more experiment to see what happens if more than one force acts on an object. Suppose an object is subject to two forces at once. Call them F_1 and F_2 . Since we are still experimenting in one dimension, the forces can act towards the right (+) or left (-) direction. Forces to the right will be considered positive and to the left negative. When both forces are applied at the same time, here is what the experiment shows:

Experiment 3: If a force F_1 produces an acceleration a_1 on an object and a force F_2 produces an acceleration a_2 on the same object, then if the force $F_1 + F_2$ acts on the object the measured acceleration is $a_1 + a_2$.

Wow, we should be grateful that nature behaves in such a simple way. The experiment indicates that if a 5 Newton force to the right and a 3 Newton force to the left are

both applied to an object, the resulting motion is the same as if a 2 Newton force to the right were applied. Forces applied in one dimension add up like real numbers. (As discussed in the next section, in two and/or three dimensions experiment shows they add up like vectors). We refer to the "sum" of all the forces on an object as the net force, and give it the label F_{net} .

Two and Three Dimensional Frictionless Environment

The extension to two and three dimensions is greatly facilitated by using the mathematics of vectors. Displacement is a vector, since displacements have the properties that vectors need to have. A displacement 40 units east plus a displacement 30 units north is the same as one displacement 50 units at an angle of $36.869\dots^\circ$ N of E. Similarly, relative velocity is a vector, since it is the time derivative of displacement.

It was shown that in the static case, forces combine as vectors. When an object moves, experiments show that forces still combine as vectors:

Experiment 3a: If two forces, \vec{F}_1 and \vec{F}_2 , act on a point particle, the resulting motion is the same as if the object were subject to the force \vec{F} where \vec{F} is the vector sum of the two forces: $\vec{F} = \vec{F}_1 + \vec{F}_2$.

The three experiments can be summarized in one simple equation:

$$\vec{F}_{net} = m\vec{a} \tag{11}$$

This equation is referred to as **Newton's Second Law of motion**. Its validity is demonstrated by the three experiments discussed above. Experiment 1 demonstrates that force and acceleration are proportional to each other and gives a method of quantifying force. Experiment 2a demonstrates that inertial mass is a scalar. Experiment 3a demonstrates that forces add according to the mathematics of vectors. Each experiment is important in establishing Newton's Second Law. Most texts agree that experiments 2a and 3a are necessary for the second law to be valid. Some texts do not include experiment 1. The idea being that we can *define* force to be proportional to acceleration. A push or pull that causes constant acceleration is the definition of a constant force. I will let you decide if a spring (or other device) can be constructed such that it will produce a constant push (or pull) without testing it dynamically.

The second law was determined using the dynamics of motion in an inertial reference frame, floating in space. It is extremely important in understanding non-relativistic motion. There is still the problem of determining the source of the forces

on an object. In this mechanics class, we will deal with forces due to friction, contact forces, and the force of gravity near a planet's surface. The general approach is to determine all the forces involved. Once the forces are known, the acceleration of the objects are known. Once the accelerations are determined, the position function $x(t)$ can be found by integration. Doing appropriate experiments we can test our understanding of the "physics" behind the interactions.

You might wonder if we are missing something in our force equation, or if there is a better way to describe motion. It turns out that differential equations are very useful in describing nature because the laws of physics often take on a simple form when expressed in terms of infinitesimal changes. This is one of the important ideas that Newton demonstrated. You might ask if there should be higher derivatives of $x(t)$ in the equation? If the force equation contained terms involving d^3x/dt^3 , one would need three initial conditions to determine $x(t)$ for times in the future. **If one believes that only the initial position, x_0 , and initial velocity, v_0 , are necessary to determine $x(t)$ for future times, then there can be at most second derivatives of $x(t)$ in the force equation.** Under these constraints, two initial conditions, forces can affect only the acceleration of a particle.

Not all quantities with magnitude and direction add according to the rules of vector addition. An example of such a quantity is rotations. Any rotation in three dimensions can be represented by an arrow with magnitude and direction: the direction is the axis of rotation, and the magnitude is the amount of rotation (e.g. in radians). Let A be a rotation about the x-axis of $\pi/2$ radians, and B be a rotation about the y-axis of $\pi/2$ radians. You can show by rotating your physics book that $\vec{A} + \vec{B} \neq \vec{B} + \vec{A}$. That forces add according to the laws of simple vector addition is remarkable.

Before we proceed with investigating different forces, we need to cover one more important law of nature: Newton's Third Law.

Newton's Third Law: symmetry in interactions

A small tack is in the vicinity of a huge strong magnet. The tack feels a strong attractive force towards the magnet, which we label as $\vec{F}_{tack \rightarrow magnet}$. Does the massive magnet feel any force due to the little tack? Yes, label this force as $\vec{F}_{magnet \rightarrow tack}$. How big is $\vec{F}_{magnet \rightarrow tack}$? Since the magnet hardly moves when the tack is attracted towards it, one might think that the magnet feels a weaker force. However, Newton realized that the magnet feels the same amount of force from the tack as the tack feels from the magnet. If the tack is attracted towards the magnet, the magnet is attracted towards the tack. These forces are on different objects and are opposite in direction: $\vec{F}_{tack \rightarrow magnet} = -\vec{F}_{magnet \rightarrow tack}$. Since acceleration = force/mass, the magnet will have

a much smaller acceleration than the tack since its mass is so much greater. This equality of interacting forces is verified by experiment and is called "Newton's Third Law":

**If object 1 exerts a force on object 2,
then object 2 exerts an equal but opposite force on object 1.**

$$\vec{F}_{12} = -\vec{F}_{21} \quad (12)$$

It was a great insight of Newton to realize there was certain symmetry in every interaction. He probably reasoned that something must be the same for each object. Interacting objects clearly can have different masses and different accelerations. The only quantity left are the forces that each object "feels". Nature is fair when it comes to interacting particles, object 2 is not preferred to object 1 when it comes to the force that each feels. Forces always come in pairs. Whenever there is a force on a particle, there must be another force acting on another particle.

Newton's Third Law applies (in some form) to every type of interaction. As simple as it may seem, it is often miss-understood. Consider the example of a book resting on a table in a room. The book feels a gravitational force due to the earth, which is its weight. What is the paired force for the book's weight? Most students answer is "the force of the table on the book". The table does exert a force on the book equal to its weight, but it is not the "reaction" force to the book's weight. The "reaction" force to the book's weight is the force on the earth due to the book. To determine the two forces that are "paired", just replace "object 1" with one object and "object 2" with the other in the statement above. "If the **earth** exerts force on the **book**, then the **book** exerts an equal but opposite force on the **earth**."

Some Simple Forces

Newton's laws of motion give us a method for determining the resulting motion when an object is subject to forces. We proceed by identifying all the forces, and then add them up using vector addition to find the objects acceleration. Once the acceleration is known at all times (or all positions) then the motion is determined. The beauty of this approach is that the forces take on a simple form. That is, the quantity that affects the acceleration of an object (the thing we are calling a force) turns out to be a simple expression of position, velocity, etc. Here, we consider three types of forces: contact forces, frictional forces, and weight (gravity near the surface of a planet). In future courses we will consider the "universal" gravitational force,

spring forces, electric and magnetic forces, and atomic/nuclear/subatomic particle interactions.

Contact Forces: By a contact force we mean the pushing or pulling caused by the touching (or physical contact) of one object on another. Someone's hand pushing on, or a rope pulling on an object are some examples. From Newton's third law, object 2 feels the same contact force from object 1 that object 1 feels from the contact force from object 2.

Consider the following example shown in the figure:

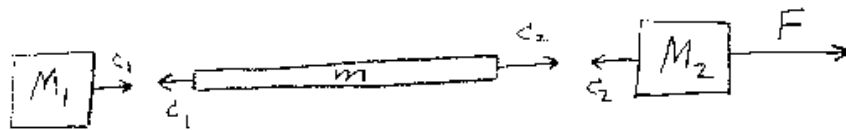
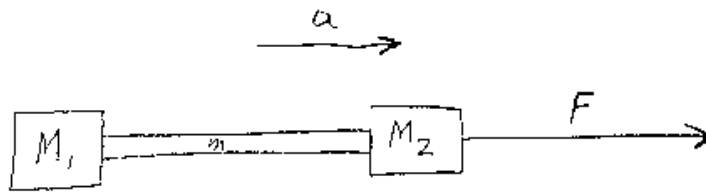
Two masses in an inertial reference frame are connected by a rope. The inertial mass of the mass on the left is M_1 , the inertial mass of the mass on the right is M_2 , and the rope connecting them has a mass of m . Someone pulls on the mass on the right with a force of F Newtons. Question: find the resulting motion, and all the contact forces.

The method of applying Newton's Laws of motion to a system of particles is: first: find the forces on each object in the system, then: the acceleration of each object is just $F_{Net}/(mass)$. For the example above, the forces on the "right mass" are F minus the force that the rope pulls to the left, which we label as c_2 . The rope feels the "reaction" force c_2 to the right and a force c_1 from the left mass. Finally, the mass on the left feels only the "reaction" force from the rope which is c_1 to the left. Summarizing:

<u>Object</u>	<u>Net Force</u>	<u>Equation of motion</u>
left mass	c_1	$c_1 = M_1 a$
rope	$c_2 - c_1$	$c_2 - c_1 = m a$
right mass	$F - c_2$	$F - c_2 = M_2 a$

Adding up the equations of motion for the various masses gives $a = F/(M_1 + M_2 + m)$. Solving for the contact forces gives: $c_1 = M_1 F/(M_1 + M_2 + m)$, and $c_2 = (M_1 + m)F/(M_1 + M_2 + m)$. If the mass m is very small compared to M_1 and M_2 , then the contact forces are approximately equal $c_1 \approx M_1 F/(M_1 + M_2)$, and $c_2 \approx M_1 F/(M_1 + M_2)$, giving $c_1 \approx c_2$. This force is called the tension in the rope, and is the same throughout for a massless rope.

Weight



$$c_1 = M_1 a \quad c_2 - c_1 = m a \quad F - c_2 = M_2 a$$

$$\begin{array}{r} c_1 = M_1 a \\ c_2 - c_1 = m a \\ F - c_2 = M_2 a \\ \hline F = (M_1 + m + M_2) a \end{array}$$

$$a = \frac{F}{M_1 + M_2 + m}$$

$$c_1 = \frac{M_1 F}{M_1 + M_2 + m}$$

$$c_2 = F - M_2 a = F - \frac{M_2 F}{M_1 + M_2 + m} = \left(\frac{M_1 + m}{M_1 + M_2 + m} \right) F$$

If $m \approx 0$, then

$$c_2 \approx \frac{M_1 F}{M_1 + M_2}$$

$$c_1 \approx \frac{M_1 F}{M_1 + M_2}$$

∴ $c_2 \approx c_1$

Weight is a force. In the Newtonian picture of gravity, the **weight** W of an object is the gravitational force on the object due to all the other matter in the universe. We will be considering the weight of objects on or near the surface of a planet (neglecting the rotation of the planet). In this case, the strongest force the object experiences is due to the planet. A remarkable property of nature is that the *motion* of all objects due to the gravitational force is not dependent on the objects inertial mass! This can be demonstrated by dropping two different objects with different inertial masses in the classroom. They both fall with the same acceleration, which we label as g . Since $F = ma$, the gravitational force on an object must be $W = mg$, where g depends only on the location of the object and m is the inertial mass of the object. An objects weight is proportional to it's inertial mass. Since $g = W/m$ is the same for all objects, if the mass is doubled, so is the objects weight.

To measure an objects weight on a planet, one can use a scale which keeps the object at rest. Since the object is not accelerating relative to the planet, the force the scale exerts on the object equals its weight. An object's mass is an intrinsic property and is the same everywhere. An objects weight W depends on its location (i.e. which planet it is on or near).

The Einstein picture of gravity is somewhat different. Being in a free falling elevator near the earth's surface "feels" the same as if you were floating in free space or in the space shuttle. In each case you are weightless. Thus, you can be near the surface of the earth and be "weightless". Likewise, if you are in a rocket ship in free space (outer space far away from any other objects) that is accelerating at $9.8M/s^2$ you feel the same as if the rocket were at rest on the earth. Thus, if you sit on a scale in a rocket that is accelerating in free space, the scale will give you a "weight" reading eventhough there are no "gravitational forces". Weight therefore is a relative quantity, and depends on the reference frame. In an inertial reference frame, everything is weightless. In a non-inertial reference frame, the force needed to keep the object at rest relative to the frame is the weight (or apparent weight) of the object. Mass (more specifically rest mass), on the other hand, is an absolute intrinsic quantity and is the same everywhere. Students interested in these philosophical topics should major in Physics. In this introductory class, we will take the Newtonian point of view for objects near the surface of a planet in which case the weight W is:

$$W = mg \tag{13}$$

Next quarter you will discover what Newton discovered, that the gravitational force between two point objects of mass m_1 and m_2 that are separated by a distance r is: $F_{gravity} = Gm_1m_2/r^2$. You will also show that the acceleration near the surface

of a spherically symmetric planet is approximately $g = Gm_{planet}/R^2$, where R is the planet's radius and $G \approx 6.67 \times 10^{-11} \text{ NM}^2/\text{Kg}^2$.

Projectile Motion

Projectile motion is often used as an example in textbooks, and is the motion of an object "flying through the air" near the surface of the earth (or any planet). The approximations that are made are 1) that the object is near enough to the surface to consider the surface as flat, 2) the acceleration due to gravity is constant (does not change with height), and 3) air friction is neglected. Usually the $+\hat{j}$ direction is taken as up, and the \hat{i} direction parallel to the surface of the earth such that the object travels in the $x - y$ plane. The objects acceleration is constant and given by $a_0 = -g\hat{j}$ for all objects. Letting $\vec{v}(t)$ represent the objects velocity vector and $\vec{r}(t)$ be the objects position vector we have:

$$\vec{v}(t) = \vec{v}_0 - gt\hat{j} \quad (14)$$

and

$$\vec{r}(t) = -\frac{gt^2}{2}\hat{j} + \vec{v}_0t + \vec{r}_0 \quad (15)$$

where \vec{v}_0 is the initial velocity and \vec{r}_0 is the initial position. If $\vec{r}_0 = 0$ and $\vec{v}_0 = v_0\cos(\theta)\hat{i} + v_0\sin(\theta)\hat{j}$ one has:

$$\vec{v}(t) = v_0\cos(\theta)\hat{i} + (v_0\sin(\theta) - gt)\hat{j} \quad (16)$$

and for the position vector, one has:

$$\vec{r}(t) = v_0\cos(\theta)t\hat{i} + (v_0\sin(\theta)t - \frac{gt^2}{2})\hat{j} \quad (17)$$

It is nice that the horizontal and vertical motions can be treated separately. This is because force is a vector and the only force acting on the particle is gravity which is in the vertical direction. The seemingly complicated two-dimensional motion is actually two simple one-dimensional motions.

In this example of projectile motion, two quantities remain constant: the acceleration ($-g\hat{j}$) and the x-component of the velocity. The x-component of the velocity is constant since there is no force in the x-direction and consequently no acceleration in the x-direction. Also note that vectors \vec{r} , \vec{v} , and \vec{a}_0 can (and usually) point in different directions.

On the earth, there is air friction which needs to be considered for an accurate calculation. Also, even in the absence of air friction, the parabolic solution above is not exactly correct. In the absence of friction, the path of a projectile near a spherical planet is elliptical.

Frictional Forces

In this class we consider "contact" frictional forces. When two surfaces touch each other, one surface exerts a force on the other (and visa-versa by Newton's third law). It is convenient to "break-up" this force into a component *perpendicular* to the surfaces (Normal Force) and a component *parallel* to the surfaces (Frictional Force). We consider two cases for the frictional force: 1) the two surfaces slide across each other (kinetic friction) and 2) the two surfaces do not slide (static friction).

Kinetic Friction

If two surfaces slide across each other, the frictional force depends primarily on two things: how much the surfaces are pushing against each other (normal force N) and the type of material that makeup the surfaces. We will show in class that the kinetic frictional force is roughly proportional to the force pushing the surfaces together (normal force N), or $F_{kinetic\ friction} \propto N$. We can change the proportional sign to an equal sign by introducing a constant: $F_{kinetic\ friction} \approx \mu_K N$. The coefficient μ_K is called the coefficient of kinetic friction and depends on the material(s) of the surfaces.

Static Friction

If the surfaces do not slide across each other, the frictional force (parallel to the surfaces) is called static friction. The static frictional force will have a magnitude necessary to keep the surfaces from sliding. If the force necessary to keep the surfaces from sliding is too great for the frictional force, then the surfaces will slip. Thus, there is a maximum value F_{Max} for the static friction: $F_{static\ friction} \leq F_{Max}$. As in the case of sliding friction, F_{Max} will depend primarily on two things: the normal force N and the type of materials that make-up the surfaces. We will also show in class the F_{Max} is roughly proportional to the normal force, $F_{Max} \propto N$. Introducing the coefficient of static friction, μ_S , we have: $F_{Max} = \mu_S N$. The static friction force will only be equal to F_{Max} just before the surfaces start slipping. If the surfaces do not slip, $F_{static\ friction}$ will be just the right amount to keep the surfaces from slipping. Thus, one usually writes that $F_{static\ friction} \leq \mu_S N$.

Summarizing we have:

$$F_{kinetic\ friction} = \mu_K N \tag{18}$$

and for static friction

$$F_{static\ friction} \leq \mu_S N \quad (19)$$

We remind the reader that the above equations are not fundamental "Laws of Nature", but rather models that approximate the forces of contact friction. The fundamental forces involved in contact friction are the electromagnetic interactions between the atoms and electrons in the two surfaces. To determine the frictional forces from these fundamental forces is complicated, and we revert to the phenomenological models described above.

It is interesting to note that in the case of kinetic friction, the net force that the surface experiences must lie on a cone. The angle that the cone makes with the normal is always $\tan^{-1}(\mu_K)$. For the case of static friction, the net force that the surface can experience must lie within a cone which makes an angle with the normal of $\tan^{-1}(\mu_S)$

Uniform Circular Motion

If an object travels with constant speed in a circle, we call the motion uniform circular motion. The uniform meaning constant speed. This motion is described by two parameters: the radius of the circle, R , and the speed of the object, v . The speed of the object is constant, but the direction of the velocity is always changing. Thus, the object does have an acceleration. We can determine the acceleration by differentiating the position **vector** twice with respect to time. For uniform circular motion, the position vector is given by:

$$\vec{r}(t) = R(\cos(\frac{vt}{R})\hat{i} + \sin(\frac{vt}{R})\hat{j}) \quad (20)$$

where \hat{i} points along the +x-direction and \hat{j} points along the +y-direction. It is also convenient to define a unit vector \hat{r} which points from the origin to the particle:

$$\hat{r}(t) = (\cos(\frac{vt}{R})\hat{i} + \sin(\frac{vt}{R})\hat{j}) \quad (21)$$

In terms of \hat{r} , the position vector \vec{r} can be written as:

$$\vec{r}(t) = R\hat{r}(t) \quad (22)$$

To find the velocity vector, we just differentiate the **vector** $\vec{r}(t)$ with respect to t:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \quad (23)$$

$$= R \frac{v}{R} \left(-\sin\left(\frac{vt}{R}\right)\hat{i} + \cos\left(\frac{vt}{R}\right)\hat{j} \right) \quad (24)$$

$$\vec{v}(t) = v \left(-\sin\left(\frac{vt}{R}\right)\hat{i} + \cos\left(\frac{vt}{R}\right)\hat{j} \right) \quad (25)$$

To find the acceleration of an object moving in uniform circular motion one needs to differentiate the velocity **vector** $\vec{v}(t)$ with respect to t :

$$\vec{a}(t) = \frac{d\vec{v}}{dt} \quad (26)$$

$$= v \frac{v}{R} \left(-\cos\left(\frac{vt}{R}\right)\hat{i} - \sin\left(\frac{vt}{R}\right)\hat{j} \right) \quad (27)$$

$$\vec{a}(t) = -\frac{v^2}{R} \hat{r} \quad (28)$$

Thus for an object moving in uniform circular motion, the magnitude of the acceleration is $|\vec{a}| = v^2/R$, and the direction of the acceleration is towards the center of the circle.

In an inertial reference frame, net force equals mass times acceleration. Thus, if an object is moving in a circle of radius R with a constant speed of v , the net force on the object must point towards the center and have a magnitude of mv^2/R .

Uniform circular motion is another case in which the three vectors \vec{r} , \vec{v} , and \vec{a} do not point in the same direction. In this case \vec{a} points in the opposite direction from \vec{r} , and \vec{v} is perpendicular to both \vec{r} and \vec{a} .

Appendix I: Vector Addition

We start with an example of vector addition. Let vector \vec{A} have a magnitude of 150 units at an angle of 30 degrees N of E, vector \vec{B} have a magnitude of 100 units at an angle of 45 degrees N of W, and vector \vec{C} have a magnitude of 200 units at an angle of 80 degrees S of E. Find the vector $\vec{D} = \vec{A} + \vec{B} + \vec{C}$, which is the sum of the three vectors. The easiest way to add these vectors is to use unit vectors (i.e. components). If \hat{i} points along the x-axis with magnitude 1, and \hat{j} points along the y-axis with magnitude 1, then

$$\begin{aligned}\vec{A} &= 150\cos(30)\hat{i} + 150\sin(30)\hat{j} \\ &= 130\hat{i} + 75\hat{j}\end{aligned}$$

Similarly,

$$\begin{aligned}\vec{B} &= -100\cos(45)\hat{i} + 100\sin(45)\hat{j} \\ &= -70.7\hat{i} + 70.7\hat{j}\end{aligned}$$

and

$$\begin{aligned}\vec{C} &= 200\cos(80)\hat{i} - 200\sin(80)\hat{j} \\ &= 34.7\hat{i} - 197\hat{j}\end{aligned}$$

To add the three vectors, one simply adds the components:

$$\begin{aligned}\vec{A} + \vec{B} + \vec{C} &= (130 - 70.7 + 34.7)\hat{i} + (75 + 70.7 - 197)\hat{j} \\ \vec{D} &= 94\hat{i} - 51.3\hat{j}\end{aligned}$$

The sum can be expressed in terms of the unit vectors, or as a magnitude and direction: $|\vec{D}| = \sqrt{94^2 + 51.3^2} = 107$ units. The vector points in the fourth quadrant, with the angle $\tan^{-1}\theta = 51.3/94$, or $\theta = 28.6^\circ$. Since \vec{D} is in the fourth quadrant, $\theta = 28.6^\circ$ S of E.

Expressing the vectors using unit (or basis) vectors is probably the most convenient way to perform operations with them. In general, if $\vec{A} = A_x\hat{i} + A_y\hat{j}$, and $\vec{B} = B_x\hat{i} + B_y\hat{j}$, then the sum $\vec{A} + \vec{B}$ equals

$$\vec{A} + \vec{B} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \quad (29)$$

for subtraction, replace the "+" with a "-". Using unit (or basis) vectors is particularly useful when adding (or subtraction) more than two vectors.

It is important to remember that **for a complete description of a vector in two dimensions, two numbers are needed**: a magnitude plus direction, two components, etc. For example, the vector \vec{D} above can be expressed as: $94\hat{i} - 51.3\hat{j}$, or 107 units at a direction 28.6° S of E, or 107 units at an angle of 331.4° clockwise from the x-axis.

It should be pointed out that all quantities with a magnitude and direction are not necessarily vectors. To be a vector, a quantity needs to have certain properties. Also, here we have limited our applications to two and three dimensional vectors. These ideas can be generalized to any number of dimensions (including infinity). For higher dimensional vector spaces, it is not always useful to think of the direction of a vector.

We end this section with an example of a quantity that has direction and magnitude, but does not have the properties required of a vector. The addition property of vectors must be commutative: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$. Rotations about an axis, which have a magnitude and direction, do not have this property. Let the arrow representing a rotation be defined in the following way: the direction is in the direction of the axis of rotation and the magnitude is equal to the amount of rotation about the axis (measured counter-clockwise). Let A be a rotation of $\pi/2$ counter-clockwise about the x-axis. The "vector" representing this rotation is $\vec{A} = (\pi/2)\hat{i}$. Let B be a rotation of $\pi/2$ counter-clockwise about the y-axis. The "vector" representing this rotation is $\vec{B} = (\pi/2)\hat{j}$. If vector addition is defined as successive rotations, $\vec{A} + \vec{B} \neq \vec{B} + \vec{A}$. That is if rotation A is first applied to an object then rotation B , the orientation of the object is different than if rotation B is first applied then rotation A . Try it out with your physics book.

There are a number of quantities in classical physics that behave as vectors: displacement, velocity, acceleration, force, the electric field and the magnetic field to name a few. You will encounter these during your first year of physics. Although they represent different physical quantities, they all add like the method described here. Vector addition pertains to many different physical quantities, it is a "universal mathematics", and its importance cannot be understated.