Mathematics Review
ME 439

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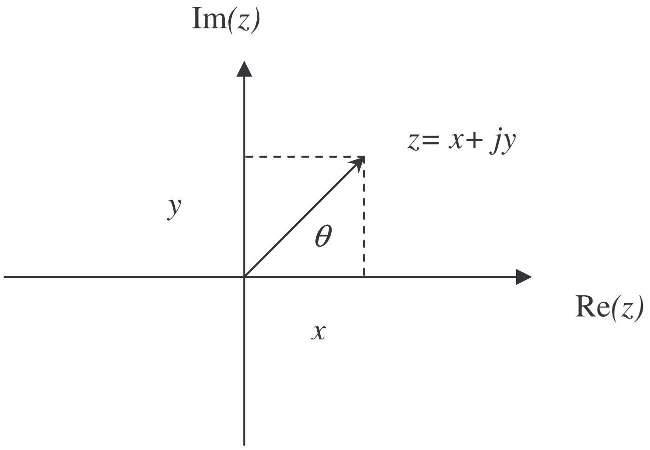
Complex Variables

Complex Plane



Definitions

$$
\begin{aligned}
&z = x + jy \\
&\mathrm{Re}(z) = x \\
&\mathrm{Im}(z) = y \\
&j = \sqrt{-1} \\
&j^2 = -1 \\
&\frac{1}{j} = -j
\end{aligned}
$$

Addition and Subtraction

$$z_1 \pm z_2 = (x_1 + x_2) \pm j(y_1 + y_2)$$

Multiplication

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

Division

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}$$

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2}\left(\frac{x_2 - jy_2}{x_2 - jy_2}\right)$$

$$\frac{z_1}{z_2} = \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + j\left(\frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2}\right)$$

Complex Conjugate

$$z^* = x - jy$$

Modulus (a.k.a. Magnitude)

$$|z| = \sqrt{x^2 + y^2}$$

Polar Form

$$x = |z|\cos\theta$$

$$y = |z|\sin\theta$$

$$\theta = Tan^{-1}\left(\frac{y}{x}\right) = Arg(z)$$

Euler's Formula

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

Useful Relationships

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Polar Form of z

$$|z|e^{j\theta} = |z|(\cos\theta + j\sin\theta) = x + jy$$
$$\theta \; (radians)$$
$$(1\; radian = \frac{180^\circ}{\pi})$$

Multiplication and Division In Polar Form

$$z_1 z_2 = |z_1||z_2|e^{j(\theta_1+\theta_2)}$$
$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1-\theta_2)}$$

Complex Conjugate in Polar Form

$$z^* = x - jy = |z|e^{-j\theta}$$
$$z\,z^* = |z|^2$$

Raising z to a Power

$$z^n = \left(|z|e^{j\theta}\right)^n = |z|^n e^{jn\theta}$$

Useful Formulae

$$|z_1 + z_2| \le |z_1| + |z_2|$$
$$|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$$
$$\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k|$$
$$\mathrm{Re}(z) = \frac{z + z^*}{2}$$
$$\mathrm{Im}(z) = \frac{z - z^*}{2j}$$
$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$$
$$|z| = |z^*|$$
$$|z_1 z_2| = |z_1||z_2|$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Residues, Poles and Zeros

Consider a complex function of *f(z)* expressed in a Laurent Series Expansion about a singular point z_o,

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z - z_o)^n = \ldots + \frac{a_{-3}}{(z - z_o)^3} + \frac{a_{-2}}{(z - z_o)^2} + \frac{a_{-1}}{(z - z_o)} + a_o + a_1(z - z_o) + a_2(z - z_o)^2 + \ldots$$

An isolated singularity z_o is a point in the *z*-plane where *f(z)* blows-up.

Residue of *f(z)*

$$\mathrm{Residue}[f(z), z_o] = \mathrm{Res}(z_o) = a_{-1}$$

Poles and Zeros

Write

$$f(z) = \frac{N(z)}{D(z)}$$

then

$N(z) = 0$ gives the Zeros of $f(z)$

$D(z) = 0$ gives the Poles of $f(z)$

The Poles are the singularities, i.e. the points which cause $f(z)$ to blow up, while the zeros are cause $f(z) = 0$.

Formulae for Finding the Residue at a Pole of Function

1) If $f(z)$ has a pole of order N at $z = z_o$, then the residue of $f(z)$ at $z = z_o$ is given by

$$a_{-1} = \frac{1}{(N-1)!} \lim_{z \to z_o} \frac{d^{N-1}}{dz^{N-1}} \left[(z - z_o)^N f(z) \right]$$

2)The Residue Theorem

$$\int_c f(z) dz = 2\pi \, j \sum_{k=1}^{n} r_k$$

where r_k is the k^{th} residue of $f(z)$

e.g. Compute the residue of

$$f(z) = \frac{z}{(z-1)^2(z+1)^3}$$

at each of it's poles

Solution

Residue at 2nd Order (N=2) pole located at z = 1,

$$r_1 = \lim_{z\to 1} \frac{d}{dz}\left[\frac{z}{(z+1)^3}\right] = -\frac{1}{16}$$

Residue at 3rd Order (N=3) pole located at z = -1,

$$r_2 = \lim_{z\to -1} \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{z}{(z-1)^3}\right] = \frac{1}{16}$$

ODE Rudiments

Consider an n^{th} order Ordinary Differential Equation (ODE) with constant coefficients as follows,

$$a_n y^n + \ldots + a_2 \ddot{y} + a_1 \dot{y} + a_o y = b_m u^m + \ldots + b_1 \dot{u} + b_o u(t) = F(t)$$

where *y(t)* is the output and *u(t)* is the input

Homogeneous ODEs $F(t) = 0$

$$a_n y_H{}^n + \ldots + a_2 \ddot{y}_H + a_1 \dot{y}_H + a_o y_H = 0$$

Assume a solution of the form

$$y_H = Ce^{\alpha t}$$

where we seek α such that the homogeneous ODE is satisfied

Plugging in the assumed solution to the homogeneous ODE leads to the characteristic equation for the eigenvalues (roots) α,

$$a_n \alpha^n + \ldots + a_2 \alpha^2 + a_1 \alpha + a_0 = 0$$

Then, the *most general form* of the homogeneous solution is the linear superposition of the individual solutions, *i.e.*

$$y_H(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} + \ldots + C_n e^{\alpha_n t}$$

Repeated Roots, *i.e.* $\alpha_1 = \alpha_2$

$$y_H(t) = C_1 e^{\alpha_1 t} + C_2 t e^{\alpha_2 t} + C_3 e^{\alpha_3 t} + \ldots + C_n e^{\alpha_n t}$$

Complex Roots, *i.e.* $\alpha_1 = \sigma + j\omega, \alpha_2 = \sigma - j\omega$

$$
\begin{aligned}
&y_H(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t} \\
&y_H(t) = C_1 e^{(\sigma + j\omega)t} + C_2 e^{(\sigma - j\omega)t} \\
&y_H(t) = e^{\sigma t}\left(C_1 e^{j\omega t} + C_2 e^{-j\omega t}\right) \\
&y_H(t) = e^{\sigma t}\left[C_1(\cos \omega t + j \sin \omega t) + C_2(\cos \omega t - j \sin \omega t)\right] \\
&y_H(t) = e^{\sigma t}\left[(C_1 + C_2)\cos \omega t + j(C_1 - C_2)\sin \omega t\right] \\
&y_H(t) = e^{\sigma t}\left[C_3 \cos \omega t + jC_4 \sin \omega t\right] \\
&y_H(t) = Ce^{\sigma t}\cos(\omega t + \phi) \\
&\phi = -Tan^{-1}\frac{C_4}{C_3} \\
&C = \sqrt{C_3^2 + C_4^2}
\end{aligned}
$$

Repeated Complex Roots

e.g. 4th order characteristic equation with

$$
\begin{aligned}
\alpha_1 &= \alpha_3 = \sigma + j\omega \\
\alpha_2 &= \alpha_4 = \sigma - j\omega
\end{aligned}
$$

here

$$y_H(t) = C_A e^{\sigma t}\cos(\omega t + \phi_A) + C_B e^{\sigma t}\cos(\omega t + \phi_B)$$

Non-Homogeneous ODEs $F(t) \neq 0$

Now,

$$y(t) = y_H(t) + y_P(t)$$

where $y_P(t)$ is the particular solution which must satisfy the original ODE as follows,

$$a_n {y_P}^n + \ldots + a_2 \ddot{y}_P + a_1 \dot{y}_P + a_o y_P = F(t)$$

How do we find $y_P(t)$?

1) Method of Undetermined Coefficients

If $F(t)$ is of the form	Then take $y_P(t)$ as
A	A
$At+B$	$At+B$
$exp(\alpha t)$	$Cexp(\alpha t)$
$\cos(\Omega t)$	$A\cos(\Omega t)+B\sin((\Omega t)$
$\sin(\Omega t)$	$A\cos(\Omega t)+B\sin((\Omega t)$

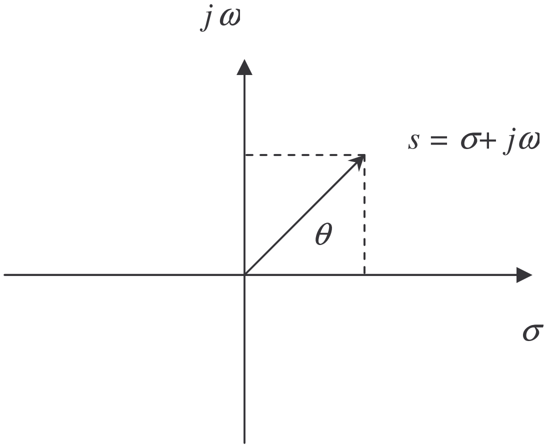
2) Variation of Parameters

This is the most general method and always works, refer to your ODE textbook for more details on VOP.

Evaluation of the Unknown Constants

The general solution $y(t) = y_H(t) + y_P(t)$ involves the unknown constants $C_1 \cdots C_n$. Use the Initial Conditions (IC's) to evaluate $C_1 \cdots C_n$ after you've built the total solution $y(t) = y_H(t) + y_P(t)$. You cannot evaluate $C_1 \cdots C_n$ until you have both the homogenous and the particular solutions in hand.

Laplace Transforms



The Complex Frequency

$$s = \sigma + j\omega$$
$$\sigma = \frac{1}{\tau} = \text{Growth Rate (Hz)}$$
$$\omega = 2\pi f = \text{Circular Frequency (rad/sec)}$$

The Laplace Transform

Let *f(t)* be a real function of time defined on $0 \le t \le \infty$, the Laplace Transform of *f(t)* is defined as

$$F(s) = \mathcal{L}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

The Laplace Transform is useful for solving forced linear ODEs with constant coefficients. We can use the Laplace Transform to solve initial value problems w/o first having to find a homogeneous solution. The Laplace Transform provides an easy method of solution for non-homogeneous problems when *F(t)* is discontinuous or impulsive. Paramount to understanding linear control theory, is the understanding of Laplace Transforms.

The Inverse Laplace Transform

$$\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \frac{1}{2\pi\, j}\int_{\alpha - j\infty}^{\alpha + j\infty} F(s)e^{st}ds$$

Laplace Transform of a Differential Equation

ODE

$$\begin{tikzcd}
\frac{dv}{dt} + \frac{1}{RC}v = v_i \arrow[r, "\text{Solution in Time Domain}"] \arrow[d, "\mathcal{L}"'] & v(t) = v_o e^{-t/RC} \\
sV(s) - V_o + \frac{V(s)}{RC} = V_i \arrow[r, "\text{Algebra in s-space}"'] & V(s) = \frac{V_o}{s + \frac{1}{RC}} \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

Laplace Transform Pairs

$F(s)$	$f(t)$	Description
1	$\delta(t)$	Unit Impulse
$\frac{A}{s}$	$A(t)=\begin{cases}0, t<0\\ A, t\geq 0\end{cases}$	Step Input
$\frac{A}{s^2}$	At	Ramp Input
$\frac{2A}{s^3}$	At^2	Parabola Input
$\frac{n!}{s^{n+1}}$	t^n	General Power Law Input
$\frac{A\omega}{s^2+\omega^2}$	$A\sin\omega t$	Sine Input
$\frac{As}{s^2+\omega^2}$	$A\cos\omega t$	Cosine Input
$sF(s)-f(0)$	$\dot{f}(t)$	First Order ODE
$s^2F(s)-sf(0)-\dot{f}(0)$	$\ddot{f}(t)$	Second Order ODE
$s^nF(s)-s^{n-1}f(0)-s^{n-2}\dot{f}(0)$ $-\ldots-f^{n-1}(0)$	$\frac{d^nf}{dt^n}$	nth Order ODE
$\frac{F(s)}{s}$	$\int f(t)dt$	Integral

$\frac{A}{\tau s+1}$	$\frac{A}{\tau}e^{-t/\tau}$	Free Response of 1st Order System
$\frac{A}{s+a}$	Ae^{-at}	
$\frac{A\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{A\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$	Free Response of an Under-damped Second Order System

Refer to your Controls Text for other important Laplace Transform Pairs.

Example

Solve the initial value problem using Laplace Transforms

$$\ddot{y} + 3\dot{y} + 2y = \sin 2t$$

subject to the initial conditions

$$y(0) = 2$$
$$\dot{y}(0) = -1$$

Taking the Laplace Transform of the ODE,

$$\mathcal{L}(\ddot{y})) + 3\mathcal{L}(\dot{y}) + 2\mathcal{L}(y) = \mathcal{L}(\sin 2t)$$
$$s^2Y(s) - 2s + 1 + 3\{sY(s) - 2\} + 2Y(s) = \frac{2}{s^2+4}$$
$$(s^2 + 3s + 2)Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{s^2+4}$$

Solving for the Laplace transform of the solution *Y(s)*

$$Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{(s^2+4)(s^2+3s+2)}$$

A partial fraction decomposition affords,

$$Y(s) = -\frac{5}{4}\frac{1}{(s+2)} + \frac{17}{5}\frac{1}{(s+1)} - \frac{1}{20}\frac{2}{(s^2+4)} - \frac{3}{20}\frac{s}{(s^2+4)}$$

Taking the inverse Laplace transform of the above expression,

$$\mathcal{L}^{-1}(Y(s)) = -\frac{5}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+2)}\right\} + \frac{17}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{2}{(s^2+4)}\right\} - \frac{3}{20}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\}$$

Hence,

$$y(t) = -\frac{5}{4}e^{-2t} + \frac{17}{5}e^{-t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t$$

Since the structure of the solution of the above example is typical of the structure obtained when solving all initial value problems for ODEs by means of Laplace transformation, a closer examination of the solution structure will help elucidate how the solution is generated.

Here, we return to the point in the solution where the ODE was transformed, the result can be rewritten as,

$$\underbrace{(s^2+3s+2)}_{\text{Transformed homogeneous ODE}} \quad Y(s) = \underbrace{2s+5}_{\substack{\text{Transformed}\\ \text{Initial Conditions}}} + \underbrace{\frac{2}{s^2+4}}_{\text{Transformed non-homogeneous term}}$$

Setting,

$$G(s) = \frac{1}{(s^2+3s+2)}$$

and denoting the transformed initial conditions as *I(s)*, and the transformed non-homogeneous term as *R(s)*, the above result can be solved for *Y(s)* and written in the form,

$$Y(s) = G(s)I(s) + G(s)R(s)$$

which illustrates how the transform *G(s)* modifies the transform of the initial conditions and the transform of the non-homogeneous term to arrive at the transform of the solution *Y(s).*

The transform *G(s)* is defined to be the ***Transfer Function***, whose name comes from the fact that when all the initial conditions are zero, so *I(s)* = 0, the only term generating a solution is the forcing function (non-homogeneous term), so $Y(s) = G(s)R(s)$ describes how the effect of the *input* is *transferred* to the *output* (the solution). Thus, we may write the Transfer Function as

$$\textit{Transfer Function} \equiv G(s) \equiv \frac{Y(s)}{R(s)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{\sin 2t\}} = \frac{\mathcal{L}\{output\}}{\mathcal{L}\{input\}}$$

Important Theorems of the Laplace Transform

The Initial Value Theorem

$$f^{r}(0)=\lim_{s\to\infty}\left\{s^{r+1}F(s)-s^{r}f(0)-s^{r-1}\dot{f}(0)-\ldots-sf^{\,r-1}(0)\right\};\quad r=0,1,\ldots n$$

In particular,

$$\begin{aligned}
f(0)&=\lim_{s\to\infty}\{sF(s)\}\\
\dot{f}(0)&=\lim_{s\to\infty}\left\{s^{2}F(s)-sf(0)\right\}\\
\ddot{f}(0)&=\lim_{s\to\infty}\left\{s^{3}F(s)-s^{2}f(0)-s\dot{f}(0)\right\}
\end{aligned}$$

The Final Value Theorem

$$f^{r}(\infty)=\lim_{s\to 0}\left\{s^{r+1}F(s)-s^{r}f(0)-s^{r-1}\dot{f}(0)-\ldots-sf^{\,r-1}(0)\right\};\quad r=0,1,\ldots n$$

In particular,

$$\begin{aligned}
f(\infty)&=\lim_{t\to\infty}f(t)=\lim_{s\to 0}\{sF(s)\}\\
\dot{f}(\infty)&=\lim_{s\to 0}\left\{s^{2}F(s)-sf(0)\right\}\\
\ddot{f}(\infty)&=\lim_{s\to 0}\left\{s^{3}F(s)-s^{2}f(0)-s\dot{f}(0)\right\}
\end{aligned}$$

The Convolution Theorem

$$(f*g)(t)=\int_{o}^{t}f(\eta)g(t-\eta)\,d\eta$$

$(f*g)\equiv$ The Convolution of the functions *f(t)* and *g(t)*

$\mathcal{L}(f*g)=F(s)\,G(s)$ is the Convolution Theorem

$$\mathcal{L}^{-1}\{F(s)G(s)\}=\int_{0}^{t}f(\eta)g(t-\eta)d\eta$$

Convolution allows us to find the time domain representation of a frequency domain transfer function.

e.g.

Find the Inverse Laplace Transform of the Transfer Function

$$T(s) = \frac{s}{\left(s^2 + a^2\right)^2}$$

Solution

First, recognize

$$\frac{s}{\left(s^2 + a^2\right)^2} = \frac{1}{\left(s^2 + a^2\right)}\frac{s}{\left(s^2 + a^2\right)} = F(s)G(s)$$

Consequently,

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{a}\sin at$$
$$\mathcal{L}^{-1}\{G(s)\} = \cos at$$

Using Convolution, we may write

$$\mathcal{L}^{-1}\left\{\frac{s}{\left(s^2 + a^2\right)^2}\right\} = \frac{1}{a}\sin at * \cos(at) = \frac{1}{a}\int_0^t \sin a\eta \cos a(t-\eta)d\eta = \frac{1}{2a}t\sin at$$

where liberal use has been made of the identities

$$\sin at = \frac{e^{jat} - e^{-jat}}{2j}$$
$$\cos a(t-\eta) = \frac{e^{j(t-\eta)} + e^{-j(t-\eta)}}{2}$$

in order to carry out the integration