Linear Algebra for Mechanical Engineers Revised (LAMER Handout)

SUPPLEMENTAL NOTES FOR MECHANICAL ENGINEERING STUDENTS

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Linear Simultaneous Equations

e.g.

$$
\begin{aligned}
5x_1 - 4x_2 + x_3 \qquad &= 0 \\
-4x_1 + 6x_2 - 4x_3 + x_4 &= 0 \\
x_1 + 6x_2 - 4x_3 + x_4 &= 0 \\
x_2 - 4x_3 + 5x_4 &= 0
\end{aligned}
$$

Matrix Notation

$$
\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Matrix Symbols

$$[A]\{x\} = \{b\}$$

where

$$
[A] = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}
$$

$$
\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}
$$

$$
\{b\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Definition

A matrix is an array of ordered numbers. A general matrix consists of *mn* numbers arranged in *m* rows and *n* columns:

$$[A]=\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{m\times n}$$

Examples

$$\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -5.3 \\ 3 & 2.1 & 6 \end{bmatrix}$$

$$\begin{Bmatrix} 6.1 & 2.2 & 3 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}$$

General Properties and Definitions of Matrices

Equivalence of Matrices

Definition

The matrices [A] and [B] are equal *iff*

(1) [A] and [B] have the same number of rows and columns
(2) All corresponding elements are equal *i.e.*

$$a_{ij}=b_{ij} \quad \text{for all } i \text{ and } j$$

e.g.

$$[A]=\begin{bmatrix}3 & 2 & 3\\ 1 & 5 & 6\\ 0 & 4 & 9\end{bmatrix}$$

$$[B]=\begin{bmatrix}3 & 2 & 3\\ 1 & 5 & 6\\ 0 & 4 & 9\end{bmatrix}$$

$$[A]=[B]$$

Transpose of a Matrix

Definition

The transpose of the $m\times n$ matrix [A], written as $[A]^T$, is obtained by interchanging the rows and columns in [A].

Example

$$[A]=\begin{bmatrix}2 & 4\\ 1 & 6\end{bmatrix}$$

$$[A]^T=\begin{bmatrix}2 & 1\\ 4 & 6\end{bmatrix}$$

If $[A] = [A]^T$

(1) the number of rows and columns in [A] are equal, $m = n$, and this defines a SQUARE MATRIX.
(2) $a_{ij} = a_{ji}$ defines a SYMMETRIC MATRIX.

Examples

$$\text{SQUARE MATRIX :}\quad [A]=\begin{bmatrix}2 & 5 & 1\\ 5 & 3 & -2\\ 1 & -2 & 4\end{bmatrix}=[A]^T$$

$$\text{SKEW SYMMETRIC :}\ [A]^T=-[A]$$

Diagonal Matrix

A diagonal matrix of order n has non-zero elements only on it's diagonal, or in other words, $a_{jk} = 0$ for all $j \neq k$.

Example

$$[A]=\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Identity (*a.k.a.* Unit) Matrix

The non-zero elements of the diagonal matrix are unity, *i.e.*

$$[I]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Null Matrix

All entries are zero, *i.e.*

$$[N]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=[0]$$

Linear Dependence and Independence

The coefficients in a set of simultaneous, linear algebraic equations can be represented as a set of row or column vectors, that is, in $[A]\{x\} = \{b\}$, the coefficient matrix $[A]$ can be written as an *augmentation* of column vectors,

$$[A]=[\{A_1\} \quad \{A_2\} \quad \cdots \quad \{A_n\}]$$

or as an *augmentation* of row vectors

$$[A]=\begin{bmatrix}\{A_1\}^T\\ \{A_2\}^T\\ \vdots\\ \{A_n\}^T\end{bmatrix}$$

Observe that all of these vectors have the same order, *i.e.* for the row vectors, they are of order 1 × *n*, and for the column vectors they are of order $n \times 1$.

In either case, the set of vectors comprising [*A*] are said to be a *linearly dependent (LD) set* if in

$$\alpha_1\{A_1\}+\alpha_2\{A_2\}+\cdots+\alpha_{n-1}\{A_{n-1}\}+\alpha_n\{A_n\}=0$$

there exists at least one value of the scalars α that is not identically equal to zero. If all of the α are equal to zero, the set of vectors is said to be *linearly independent (LI)*, and the vectors constitute a linearly independent set. Note, that if a set of vectors is not *LI*, that set must be *LD*.

Example

Consider the matrix

$$[A]=\begin{bmatrix}3 & -2 & 0\\ -2 & 4 & -1\\ 0 & -1 & 6\end{bmatrix}$$

This matrix may be represented as three row vectors

$$\begin{aligned}\{A_1\}^T &=\{3 \quad \text{-}2 \quad 0\}\\ \{A_2\}^T &=\{-2 \quad 4 \quad \text{-}1\}\\ \{A_3\}^T &=\{0 \quad \text{-}1 \quad 6\}\end{aligned}$$

Checking to see if the three vectors above form a linearly independent set, we construct

$$\alpha_1\{A_1\}^T+\alpha_2\{A_2\}^T+\alpha_3\{A_3\}^T=\{0 \quad 0 \quad 0\}^T$$

Expanding the above leads to the following set of three equations in three unknowns

$$\begin{aligned}&3\alpha_1-2\alpha_2=0\\&-2\alpha_1+4\alpha_2-\alpha_3=0\\&-\alpha_2+6\alpha_3=0\end{aligned}$$

From whence it is noted that the third of these affords

$$\alpha_3=\frac{\alpha_2}{6}$$

while from the first,

$$\alpha_1=\frac{2\alpha_2}{3}$$

Using this information in the second equation,

$$-2\left(\frac{2\alpha_2}{3}\right)+4\alpha_2-\left(\frac{\alpha_2}{6}\right)=0$$

Thus,

$$\alpha_2=0$$

and since

$$\alpha_1=\alpha_2=\alpha_3=0$$

the set of three row vectors form a linearly independent (LI) set.

Example

The matrix $[A]$ given by

$$[A]=\begin{bmatrix}4&-2&0\\2&-1&0\\0&2&3\end{bmatrix}$$

can be represented as a set of three column vectors

$$\{A_1\}=\begin{Bmatrix}4\\2\\0\end{Bmatrix},\quad \{A_2\}=\begin{Bmatrix}-2\\-1\\2\end{Bmatrix},\quad \{A_3\}=\begin{Bmatrix}0\\0\\3\end{Bmatrix}$$

Checking for *LI* or *LD*, we formulate,

$$\alpha_1\{A_1\}+\alpha_2\{A_2\}+\alpha_3\{A_3\}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

which provides

$$\begin{aligned}4\alpha_1-2\alpha_2&=0\\2\alpha_1-\alpha_2&=0\\2\alpha_2+3\alpha_3&=0\end{aligned}$$

Now, solving gives

$$\alpha_3=-2\frac{\alpha_2}{3}=-4\frac{\alpha_1}{3}$$

There is no guarantee that the value of any coefficient α is identically zero. In fact, the solution for the above α may be expressed as

$$\begin{Bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{Bmatrix}=\beta\begin{Bmatrix}1\\2\\-4/3\end{Bmatrix}$$

where the parameter β may be any scalar, not necessarily zero. The column vectors in the matrix [*A*] of this example, thus from a linearly dependent (*LD*) set, and any larger set containing these vectors as a subset would form a linearly dependent set.

Matrix Addition

$$[C] = [A] + [B]$$

iff

(1) [A] and [B] have the same number of rows and columns
(2) $c_{ij} = a_{ij} + b_{ij}$

Summary of Rules for Combined Matrix Operations

$$
\begin{aligned}
&[A]+[B]=[B]+[A] \\
&[A]+[0]=[A] \\
&[A]+[B]+[C]=[A]+\big[[B]+[C]\big]=\big[[A]+[B]\big]+[C] \\
&\alpha\big[[A]\pm[B]\big]=\alpha[A]\pm\alpha[B] \\
&[A]+[A]=[A]\big[[I]+[I]\big]=2[A][I]=2[A] \\
&\big[[A]\pm[B]\big]^T=[A]^T\pm[B]^T
\end{aligned}
$$

Example

$$[A]=\begin{bmatrix} 2 & 1 & 1 \\ 5 & 3 & 0 \end{bmatrix}$$

$$[B]=\begin{bmatrix} 3 & 1 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

$$[C]=[A]+[B]=\begin{bmatrix} 5 & 2 & 3 \\ 7 & 7 & 1 \end{bmatrix}$$

Multiplication of Matrices

Two matrices $[A]$ and $[B]$ can be multiplied to obtain

$$\underbrace{[C]}_{p\times q}=\underbrace{[A]}_{p\times m}\underbrace{[B]}_{m\times q}$$

iff

(1) the *number of columns* in [*A*] is equal to the *number of rows* in [*B*]

(2) then

$$[C] = c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

Example

$$[A] = \begin{bmatrix} 5 & 3 & 1 \\ 4 & 6 & 2 \\ 10 & 3 & 4 \end{bmatrix}$$

$$[B] = \begin{bmatrix} 1 & 5 \\ 2 & 4 \\ 3 & 2 \end{bmatrix}$$

$$[C] = [A][B] = \begin{bmatrix} 14 & 39 \\ 22 & 48 \\ 28 & 70 \end{bmatrix}$$

<u>Properties of Matrix Multiplication</u>

$$[A][I] = [I][A] = [A]$$

$$\alpha[A]\,[B] = [\alpha\, A][B] = [A][\alpha\, B]$$

$$\alpha[A][\,[B]+[C]\,] = \alpha[A][B] + \alpha[A][C]$$

$$\alpha[\,[A]+[B]\,][C] = \alpha[A][C] + \alpha[B][C]$$

$$[\,[A]+[B]\,]^2 = [A]^2 + 2[A]\,[B] + [B]^2$$

$$[\,[A]-[B]\,]^2 = [A]^2 - 2[A]\,[B] + [B]^2$$

$$[\,[A]-[B]\,][\,[A]+[B]\,] = [A]^2 + [A]\,[B] - [B][A] + [B]^2$$

$$\alpha[A][B][C] = [\alpha\, A][B][C] = [A][\alpha\, B][C] = [A][B][\alpha\, C]$$

$$[\,[A][B]\,]^T = [B]^T\,[A]^T$$

$$[\,[A][B][C]\,]^T = [C]^T\,[B]^T\,[A]^T$$

The Determinant of a Matrix

The determinant of an $n \times n$ matrix $[A]$ is denoted by det$[A]$ or $|A|$ is defined by

$$\det[A] = |A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det[A]_{ij}$$

where $[A]_{1j}$ is the $(n-1) \times (n-1)$ matrix obtained by eliminating the 1st row and the j^{th} column from the matrix $[A]$.

Example

$$[A] = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\det[A] = |A| = 8$$

Note: det($[A][B]$)=det$[A]$det$[B]$ or $|AB|=|A||B|$, and so on.

Definition

Let $[A]$ be an $n \times n$ matrix. Let $[M]_{ij}$ be the $(n-1) \times (n-1)$ sub-matrix of $[A]$ obtained by deleting the i^{th} row and j^{th} column of $[A]$. The determinant of $|M_{ij}|$ is called the MINOR of a_{ij}.

Example

$$[A] = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$$

$$|M_{12}| = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34$$

$$|M_{23}| = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10$$

Linear Independence Revisited

Now, after having introduced the definition of a matrix determinant, let us consider a set S of n-dimensional column vectors for the case where $m = n$

$$S = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$$

Checking for LI,

$$\alpha_1 \begin{Bmatrix} u_{11} \\ \vdots \\ u_{n1} \end{Bmatrix} + \alpha_2 \begin{Bmatrix} u_{12} \\ \vdots \\ u_{n2} \end{Bmatrix} + \cdots + \alpha_n \begin{Bmatrix} u_{1n} \\ \vdots \\ u_{nm} \end{Bmatrix} = [U]\{\alpha\} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \end{Bmatrix}$$

where

$$\{\alpha\} = \{\alpha_1, \ldots, \alpha_n\}^T$$

The system of algebraic equations

$$[U]\{\alpha\} = \{0\}$$

has a non-trivial (non-zero) solution $\{\alpha\}$ *iff*

$$\det[U] = 0 \Rightarrow S \text{ is LD (Linear Dependent)}$$
$$\det[U] \neq 0 \Rightarrow S \text{ is LI (Linear Independent)}$$

Singular Matrices

A matrix $[A]$ is singular if $\det[A] = 0$.

Ill-conditioned Matrices

A matrix $[A]$ is ill-conditioned if $\det[A] \approx 0$.

Rank of a Matrix

The rank of a matrix $[A]$, written as $r([A])$, is the largest value of r for which there exists an $r \times r$ sub-matrix of $[A]$ that is non-singular.

Example

The matrix

$$[A]=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}$$

is singular because $|A| = 0$. This matrix cannot have a rank of 3, *i.e.* $r([A]) \neq 3$. However, it is easy to find a 2×2 sub-matrix, for instance

$$[S]=\begin{bmatrix}1 & 2\\ 4 & 5\end{bmatrix}$$

where $|S| \neq 0$, here $|S| = -3$, and thus the rank of the matrix $[A]$ is $r = 2$.

Cofactor of a Matrix

Definition

Let $[A]$ be an $n \times n$ matrix. The COFACTOR A_{ij} of a_{ij} is defined as

$$\text{cof}[A] = A_{ij} = (-1)^{i+j}\left|M_{ij}\right|$$

Adjoint of a Matrix

Definition

Let $[A]=[a_{ij}]$ be an $n \times n$ matrix. The ADJOINT of $[A]$ is the matrix whose (i,j) entry is the COFACTOR A_{ji} of a_{ji}

Thus,

$$\text{adj}[A] = (\text{cof}[A])^T = \begin{bmatrix}A_{11} & A_{21} & \cdots & A_{n1}\\ A_{12} & A_{22} & & A_{N2}\\ A_{13} & \vdots & & \vdots\\ A_{1n} & A_{2n} & \cdots & A_{nn}\end{bmatrix}$$

Example

$$[A]=\begin{bmatrix}3 & -2 & 1\\5 & 6 & 2\\1 & 0 & -3\end{bmatrix}$$

$$e.g.\quad A_{12}=-6$$

$$\text{adj}[A]=\begin{bmatrix}-18 & -6 & -10\\17 & -10 & -1\\-6 & -2 & 28\end{bmatrix}$$

Inverse of a Matrix

The inverse of an $n \times n$ matrix $[A]$ is denoted by inv$[A]$=$[A]^{-1}$ and is an $n \times n$ matrix such that

$$[A][A]^{-1}=[A]^{-1}[A]=[I]$$

$$[A]^{-1}=\frac{\text{adj}[A]}{|A|}=\frac{(\text{cof}[A])^{T}}{|A|}$$

Example

$$[A]=\begin{bmatrix}3 & -2 & 1\\5 & 6 & 2\\1 & 0 & -3\end{bmatrix}$$

$$[A]^{-1}=\frac{1}{-94}\begin{bmatrix}-18 & -6 & -10\\17 & -10 & -1\\-6 & -2 & 28\end{bmatrix}$$

Inverse of a 2 × 2 Matrix

Consider the following generic 2 × 2 matrix,

$$[A]=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$[A]^{-1}=inv[A]=\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}=\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad-bc}=\begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

Inverse of a 3 × 3 Matrix

$$[A]=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$[A]^{-1}=\frac{1}{|A|}\begin{bmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ +\begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

Cramer's Rule

Consider,

$$\begin{aligned} a_1x+b_1y+c_1z &= d_1 \\ a_2x+b_2y+c_2z &= d_2 \\ a_3x+b_3y+c_3z &= d_3 \end{aligned}$$

If

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

Then, the system above has a unique solution given by

$$x = \frac{1}{D}\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} \qquad y = \frac{1}{D}\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad z = \frac{1}{D}\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Example, let us solve for the unknowns of a 2×2 system, using the 2×2 matrix inversion rule of thumb and Cramer's Rule.

Consider,

$$\begin{aligned} 3x - 2y &= 7 \\ 4x + 5y &= 2 \end{aligned}$$

Using the 2×2 matrix inversion rule of thumb,

$$\begin{aligned} &[A]\{u\} = \{r\} \\ &\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 7 \\ 2 \end{Bmatrix} \\ &\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1} \begin{Bmatrix} 7 \\ 2 \end{Bmatrix} \end{aligned}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \frac{\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}}{3(5)-4(-2)} \begin{Bmatrix} 7 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \frac{1}{23} \begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix} \begin{Bmatrix} 7 \\ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \frac{1}{23} \begin{Bmatrix} 5(7)+2(2) \\ -4(7)+3(2) \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \frac{1}{23} \begin{Bmatrix} 39 \\ -22 \end{Bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 39/23 \\ -22/23 \end{Bmatrix}$$

Using Cramer's Rule for this 2×2 system,

$$x = \frac{\begin{vmatrix} 7 & -2 \\ 2 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix}} = \frac{7(5)-(-2)2}{3(5)-(-2)4} = \frac{39}{23}$$

$$y = \frac{\begin{vmatrix} 3 & 7 \\ 4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix}} = \frac{3(2)-(7)4}{3(5)-(-2)4} = -\frac{22}{23}$$

Upper and Lower Triangular Matrices

Let $[A]$ be an $n \times n$ matrix

(1) $[A]$ is UPPER TRIANGULAR if $a_{ij} = 0$ for $i > j$

e.g.

$$[U]=\begin{bmatrix}1 & 3 & 3\\ 0 & 3 & 5\\ 0 & 0 & 2\end{bmatrix}$$

(2) [A] is LOWER TRIANGULAR if $a_{ij} = 0$ for $i < j$

e.g.

$$[L]=\begin{bmatrix}1 & 0 & 0\\ 2 & 3 & 0\\ 3 & 5 & 2\end{bmatrix}$$

Trace of a Matrix

In both of the above cases, det[A] = tr[A]= 1×3×2=6, where tr[A] denotes the TRACE of [A],which in general is given by

$$\operatorname{tr}[A]=\sum a_{ii}$$

Singular and Non-singular Matrices

Definition

An $n \times n$ matrix is called NON-SINGULAR if inv[A] exists (or in other words [A] is invertible). Otherwise, [A] is SINGULAR.

Theorem

The inverse of a matrix, if it exists, is unique.

Theorem

If [A] and [B] are both NON-SINGULAR, then [A][B] is NON-SINGULAR and inv([A][B]) = inv[B]inv[A].

Summary of Useful Results Involving inv[A]

$$[\alpha A]=\frac{1}{\alpha}[A]^{-1}$$
$$\left([A]^{n}\right)^{-1}=\left([A]^{-1}\right)^{n}$$

Other Useful Relationships

$$\left([A]^T\right)^{-1} = \left([A]^{-1}\right)^T$$
$$\left([A]^{-1}\right)^{-1} = [A]$$

Positive Definite and Negative Definite Matrices

A quadratic form $\{x\}^T[A]\{x\}$ is classified as POSITIVE DEFINITE (i.e. "definitely positive") if $\{x\}^T[A]\{x\} > 0$ for all $\{x\} \neq \{0\}$, and is NEGATIVE DEFINITE if $\{x\}^T[A]\{x\} < 0$ for all $\{x\} \neq \{0\}$. Likewise, $[A]$ is classified as positive (negative) definite if the quadratic form $\{x\}^T[A]\{x\}$ is positive (negative) definite.

The Eigenvalue Problem

Matrix equations of the form

$$[A]\{x\} = \lambda\{x\}$$

occur frequently in engineering analyses. Consider, for instance, that the variables of interest in the transient analysis of a linear system are x, y and z, and that they are related by three linear, simultaneous, non-homogeneous ordinary differential equations with constant coefficients:

$$\begin{aligned}
&\frac{dx}{dt} + 3x - 2y - 4z = 12 \\
&-x + \frac{dy}{dt} + 2y = 0 \\
&-2x + y + \frac{dz}{dt} = 0
\end{aligned}$$

These may be solved for the derivatives

$$\begin{aligned}
\frac{dx}{dt} &= -3x + 2y + 4z + 12 \\
\frac{dy}{dt} &= x - 2y \\
\frac{dz}{dt} &= 2x - y
\end{aligned}$$

and then cast into state-space (matrix) form as follows

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} = \begin{bmatrix} -3 & 2 & 4 \\ 1 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} + \begin{Bmatrix} 12 \\ 0 \\ 0 \end{Bmatrix}$$

which compactly reads

$$\{\dot{x}\} = [A]\{x\} + \{b\}$$

where

$$[A] = \begin{bmatrix} -3 & 2 & 4 \\ 1 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$

and the "forcing functions" are given by

$$\{b\} = \begin{Bmatrix} 12 \\ 0 \\ 0 \end{Bmatrix}$$

The solution to the system of ODEs begins with the determination of the so-called complementary function. The procedure is to make the set of ODEs homogeneous and then, knowing that exponential solutions exist, assume that the complimentary function is in the form

$$x(t) = \gamma\, e^{\lambda t}$$

where $\gamma$is an arbitrary constant. Thus, in

$$\{\dot{x}_c\} = [A]\{x_c\}$$

we take

$$\{x_c\} = \{C\}e^{\lambda t}$$

where $\{C\}$ is a 3×1 column vector of arbitrary constants. Then with

$$\{\dot{x}_c\} = \lambda\{C\}e^{\lambda t}$$

it is trivial to show that

$$\lambda\{x_c\} = [A]\{x_c\}$$

which is in the form of the eigenvalue problem

$$[A]\{x\} = \lambda\{x\}$$

and where the values of the various λ must then be determined. This illustration describes what is commonly referred to as the *eigenvalue* or *characteristic value problem*. It occurs frequently in engineering analysis in all disciplines. It would be a mistake, however, to believe that the eigenvalue problem comes forth only from a set of differential equations. As an example of this, consider the following set of simultaneous algebraic equations

$$[A]\{x\} = \{y\}$$

in which the column vector of constants $\{y\}$ may have been derived from a linear transformation of the form

$$\{y\} = \lambda\{x\}$$

Consequently,

$$[A]\{x\} = \lambda\{x\}$$

is in the form of the eigenvalue problem and the scalar multiplier λ is to be determined.

Eigenvalues

The eigenvalue problem stated as

$$[A]\{x\} = \lambda\{x\}$$

can be rearranged into the form

$$[A - \lambda I]\{x\} = \{0\}$$

where the matrix

$$[K] = [A - \lambda I]$$

is defined to be the *characteristic matrix* of the matrix [*A*] and for the homogenous set of equations represented by

$$[A - \lambda I]\{x\} = \{0\}$$

non-trivial solutions can be obtained *if and only if*

$$\det K = |K| = |A - \lambda I| = 0$$

The determinant of [*K*], given by |*K*| is referred to as the *characteristic determinant* or the *characteristic function* of the matrix [*A*], and is clearly a polynomial in λ,

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_1 \lambda + \alpha_0$$

The polynomial $p(\lambda)$ is known as the *characteristic polynomial* of the matrix [*A*], and when it is set equal to zero in accordance with the requirement that |*K*|=0 for non-trivial solutions to exist, we obtain

$$p(\lambda) = \alpha_n \lambda^n + \alpha_{n-1}\lambda^{n-1} + \alpha_{n-2}\lambda^{n-2} + \cdots + \alpha_1 \lambda + \alpha_0 = 0$$

which we deem to be the *characteristic equation* of the matrix [*A*]. The λ values which satisfy the *characteristic equation* are the roots of $p(\lambda) = 0$, and are labeled the *characteristic values* or *eigenvalues*. Eigenvalues (characteristic roots) are in general, complex numbers and may indeed be repeated.

Example

Consider the matrix [*A*] given below

$$[A] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

and formulate the characteristic matrix [*K*] as follows

$$[K]=[A-\lambda I]=\begin{bmatrix}-\lambda & 1 & 0\\ 0 & -\lambda & 1\\ -6 & -11 & -(6+\lambda)\end{bmatrix}$$

Evaluating the determinant of [*K*] and setting it equal to zero yields the characteristic equation

$$p(\lambda)=\lambda^3+6\lambda^2+11\lambda+6=0$$

The roots of the above cubic equation are readily found, and ordering them with the largest first are given by

$$\lambda_1=-1$$
$$\lambda_2=-2$$
$$\lambda_3=-3$$

where we see that the eigenvalues of the matrix [*A*] are real, separate and distinct.

Eigenvectors for Non-repeated Eigenvalues

Every eigenvalue leads to a solution of the eigenvalue problem given by

$$[A]\{x\}=\lambda\{x\}$$

If, in addition, the eigenvalues are non-repeated, then

$$[A-\lambda_k I]\{x_k\}=\{0\};\quad (k=1,2,3,\ldots,n)$$

where [*A*] is $n \times n$ and where the λ_k are the *n* characteristic roots (eigenvalues) of [*A*]. Because each $\{x\}$ is $n \times 1$, we have above, a set of *n* linear, simultaneous homogeneous algebraic equations for which a non-trivial solution exists *if and only if*

$$\det[A-\lambda_k I]=0$$

The solution vector $\{x_k\}$ for each eigenvalue is called a characteristic vector or eigenvector and the set of *n* eigenvectors that are all $n \times 1$ constitute a linearly independent set if all the eigenvalues are distinct or unequal.

The actual determination of the eigenvectors is accomplished by placing the eigenvalue associated with the eigenvector into

$$[A-\lambda_k I]\{x_k\}=\{0\}; \quad (k=1,2,3,\ldots,n)$$

and solving the resulting system of equations. This process is illustrated in the example below.

Example

Revisiting the previous example we found that for the matrix [*A*]

$$[A]=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

the three real, distinct eigenvalues were

$$\begin{aligned} \lambda_1 &= -1 \\ \lambda_2 &= -2 \\ \lambda_3 &= -3 \end{aligned}$$

Now the eigenvector $\{x_1\}$ can be found by using $\lambda_1 = -1$ in the set of three homogenous, linear simultaneous equations given by

$$[A-\lambda I]\{x\}=\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -6 & -11 & -(6+\lambda) \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

affording

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{bmatrix}\begin{Bmatrix} x \\ y \\ z \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} x + y &= 0 \\ y + z &= 0 \\ -6x - 11y - 5z &= 0 \end{aligned}$$

consequently,

$$\begin{aligned} x &= -y \\ z &= -y \end{aligned}$$

Thus, with an arbitrarily convenient selection of $y = 1$, the first eigenvector is given by

$$\{x_1\} = \alpha_1 \begin{Bmatrix} -1 \\ 1 \\ -1 \end{Bmatrix}$$

For λ_2= -2, we proceed as before, starting with

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -6 & -11 & -4 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

providing

$$\begin{aligned} 2x + y &= 0 \\ 2y + z &= 0 \\ -6x - 11y - 4z &= 0 \end{aligned}$$

from whence, we see we have another linearly dependent system wherein

$$\begin{aligned} x &= -\frac{y}{2} \\ z &= -2y \end{aligned}$$

so that with the arbitrary selection of $y = 1$, the second eigenvector becomes

$$\{x_2\} = \alpha_2 \begin{Bmatrix} -1/2 \\ 1 \\ -2 \end{Bmatrix}$$

Finally, with λ_3= -3,

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -6 & -11 & -3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

gives the following

$$\begin{aligned} &3x + y = 0 \\ &3y + z = 0 \\ &-6x - 11y - 3z = 0 \end{aligned}$$

here

$$\begin{aligned} x &= -\frac{y}{3} \\ z &= -3y \end{aligned}$$

so that with the arbitrary selection of $y = 1$, the third eigenvector becomes

$$\{x_3\} = \alpha_3 \begin{Bmatrix} -1/3 \\ 1 \\ -3 \end{Bmatrix}$$

The reader may verify that the three eigenvectors given above, do indeed form a linearly independent (*LI*) set.

Eigenvectors for Repeated Eigenvalues

When eigenvalues of the matrix [*A*] are repeated with a multiplicity of *r*, some of the eigenvalues may be linearly dependent on the others. Guidance on the number of linearly independent eigenvectors can be obtained from the rank of the matrix [*A*]. It turns out that a set of simultaneous, linear, homogeneous algebraic equations, if consistent, produces a unique solution if the rank of the $n \times n$ coefficient matrix is equal to its order (size). If the rank of the coefficient matrix is less than its order, an infinite number of solutions is produced.

To ascertain how many *LI* eigenvectors are affiliated with each repeated eigenvalue, it is necessary to examine the rank of the matrix [*K*] = [*A* - λI]. The first step is to form [*K*] with the repeated eigenvalue inserted. Then the rank of [*K*] is determined and the number of *LI* eigenvectors associated with the repeated eigenvalue is equal to the difference between the order of [*K*] and the rank of [*K*], *i.e.* $n - r$.

Example

For the matrix

$$[A] = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

the characteristic matrix is

$$[K] = [A - \lambda I] = \begin{bmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 0 & 0 & (2-\lambda) \end{bmatrix}$$

The characteristic equation is thus

$$p(\lambda) = \det[K] = \lambda^2(2-\lambda) - 4(2-\lambda) = (2-\lambda)(\lambda^2 - 4) = 0$$

and this yields three eigenvalues,

$$\lambda_1 = \lambda_2 = 2$$
$$\lambda_3 = -2$$

With insertion of the first two eigenvalues, $[A - \lambda I]\{x\}$ becomes

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

and for this particular [*K*], which is 3 × 3, ($n = 3$), it is easy to show that the rank of [*K*] is r ([*K*]) = 1. Thus, with $n - r = 3 - 1 = 2$, two *LI* eigenvectors are obtained. Expansion of the foregoing matrix representation yields two *LD* equations

$$-2x + 2y = 0$$
$$2x - 2y = 0$$

which, from either, provides $x = y$, meaning that z can be arbitrary (which shows that an infinite number of solutions exist).

The two *LI* eigenvectors can be obtained by setting $x = 1$, for instance , and letting $z = 0$ or by arbitrarily setting $z = 1$ and calling $x = y = 0$, say. In this scenario, the two eigenvectors are:

$$\{x_1\} = \alpha_1 \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}, \qquad \{x_2\} = \alpha_2 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

which are clearly *LI*.

The third eigenvalue $\lambda_3 = -2$ inserted into the characteristic matrix affords the following simultaneous, linear homogeneous equations

$$[A - \lambda I]\{x\} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

or in expanded form

$$\begin{aligned} 2x + 2y &= 0 \\ 2x + 2y &= 0 \\ 4z &= 0 \end{aligned}$$

Now, $z = 0$ and $x = -y$, so that another *LI* eigenvector comes forth as

$$\{x_3\} = \alpha_3 \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$$

NUMERICAL LINEAR ALGEBRA USING EXCEL

The Microsoft Excel matrix functions available are the following:

MDETERM(array)	Returns the matrix determinant of an array
MINVERSE(array)	Returns the inverse of the matrix of an array
MMULT(arrayA, arrayB)	Returns the matrix product
TRANSPOSE(array)	Returns the transpose of an array

Note: except for the MDETERM( ) function, the above array functions must be entered into the EXCEL spreadsheet using CTRL + SHIFT + ENTER

Example

Consider the linear algebraic system of equations given by:

$$
\begin{aligned}
&2x + 3y - 2z = 15\\
&3x - 2y + 2z = -2\\
&4x - y + 3z = 2
\end{aligned}
$$

Hence, if we define

$$[A] = \begin{bmatrix} 2 & 3 & -2 \\ 3 & -2 & 2 \\ 4 & -1 & 3 \end{bmatrix}$$

$$\{R\} = \begin{Bmatrix} 15 \\ -2 \\ 2 \end{Bmatrix}$$

Then the system is written compactly as

$$[A]\{x\} = \{R\}$$

and we seek the solution

$$\{x\} = [A]^{-1}\{R\} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix}$$

The above problem is solved using EXCEL as shown on the following page.

	A	B	C	D	E	F	G
1	Matrix of Coeffs [A]				Right Hand Side Vector {R}		
2	2	3	-2		15		
3	3	-2	2		-2		
4	4	-1	3		2		
5							
6	Inverse of [A] = Inv[A}				Solution Vector {x} = Inv[A]{R}		
7	0.190476	0.333333	-0.095238		2	x	
8	0.047619	-0.666667	0.47619		3	y	
9	-0.238095	-0.666667	0.619048		-1	z	
10							
11	det[A]	-21					
12							
13	tran[A]						
14	2	3	4				
15	3	-2	-1				
16	-2	2	3				
17							

To construct the above spreadsheet, follow the four steps outlined below:

a) In cells A2:C4 the coefficients of matrix $[A]$ are entered.

b) In cells E2:E4 the coefficients of right hand side vector $\{R\}$ are entered.

c) With the range of cells A7:C9 highlighted, enter the formula =MINVERSE(A2:C4) followed by CTRL+SHIFT+ENTER.

d) Finally, the solution vector $\{x\}$ is entered in the range of highlighted cells E7:E9 using the formula =MMULT(A7:C9,E7:E9) followed by CTRL+SHIFT+ENTER.

Continuing, say we wanted to compute the determinant $|A|$ and the transpose $[A]^T$.

Referring to the above EXCEL sheet, this is easily done using the following:

e) To compute det$[A]$, enter the formula =MDETERM(B3:D5) in cell B11 to return the value of –21 as the determinant of matrix $[A]$ (Again note: you do not have to follow the formula with the CTRL+SHIFT+ENTER combo since det$[A]$ is a scalar value).

f) To compute the transpose of $[A]$ highlight the range of cells A14:C16 and type in the formula =TRANSPOSE(B3:D5) to compute the elements of tran$[A]$ as shown above

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INTRODUCTION TO MATLAB

MATLAB Environment

The MATLAB programming environment provides the user with an interactive work space, in which the user enters commands and view results. After launching MATLAB from the Desktop Icon (or via Start -> Programs -> MATLAB), commands are entered at the >> prompt. A number of Operating System (OS) level control-type commands are available at the MATLAB >> prompt, including:

>> `cd`	Changes the current working directory
>> `delete`	Delete a file
>> `diary`	Saves your MATLAB session as a text file
>> `dir`	List the contents of the current directory
>> `!`	Execute an OS command
>> `quit`	Exit MATLAB

MATLAB Variables and Statements

Scalar, vector and matrix quantities can be easily defined in MATLAB using assignment statements. Some examples follow:

`a= 14.7`	defines a scalar
`r = [1 2 3 4]`	defines a row vector
`c = [1.2 3.2 7.5 5.6]’`	defines a column vector (prime means transpose)
`b = [1 2 3; 4 5 6;7 8 9]`	defines a 3 x 3 matrix, rows are delineated by a ; (Is this example singular ?)

Array elements may be referenced directly, *e.g.* `c(2) = 3.2`, or `b(3,2) = 8`. Built into MATLAB are the usual mathematical functions, such as `sin, cos, tan, atan`, *etc.* as well as everybody’s favorite constant `pi`.

MATLAB Logic

The MATLAB programming language provides loops, branches and functions, *akin* to any other high level modular programming language*, i.e.* `for ,if else.`

MATLAB Script Files and Functions

Script files and functions are *batch* files which contain a series of >> input command lines appended together. These files are given the extension .m and are referred to in MATLAB slang as “**M-files**”. Scripts are created in the user’s text editor of choice and executed by typing the name of the script at the >> prompt.

The generic structure of a `function` declaration in MATLAB is given below:

```
function[output variables] = function_name(input variables)
```

MATLAB Plotting Routine

Plotting your results is achieved with the command

```
>> plot(t, v), xlabel(’ x axis name’), ylabel(’ y axis
name’), title(’ Welcome to MATLAB for Dummies’)
```

Example 1) “Use of Matrix Notation”

We can use matrix operations to reduce the number of lines to be typed into the MATLAB derivative function file. Consider, the following 2nd order ODE which is the classic spring-mass-damper model describing the motion of a mass connected to a spring, and viscous damping.

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

where *x* is the displacement of the mass, *c* is the linear damping coefficient, *k* is the linear spring constant and *f(t)* is the forcing function. The above system can be placed into state-space (Cauchy) form by defining the following quantities:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m} f(t) - \frac{k}{m} - \frac{c}{m} x_2 \end{aligned}$$

In state-space form the above system reads

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \dfrac{1}{m} \end{Bmatrix} f(t)$$

Which in compact notation is given by

$$\{\dot{x}\} = [A]\{x\} + \{b\} f(t)$$

The following MATLAB function file illustrates how easily it is to use matrix operations within MATLAB. Here *m* = 1 kg, *c* = 2 N-s/m , *k* = 5 N/m and *f* = 10 N (a constant input step function).

```
Function xdot = msd(t,x)
% function file for mass with spring and damping
```

```
% position is first variable, velocity is second variable
global c f k m
A = [0, 1; -k/m, -c/m];
B = [0; 1/m];
xdot = A*x+B*f;
```

The corresponding driver file is given by

```
% driver for k-m-c example
global c f k m
m = 1; c = 2; k = 5; f = 10;
[t,x]= ode23(‘msd’, [0,5],[0,0]);
plot(t,x),  xlabel(‘Time  (sec)’),  ylabel(‘Displacement  (m)
and Velocity (m/s)’)
gtext(‘Displacement’)
gtext(‘Velocity’)
```

Suppose we save the above script as `kmcdriver.m`, then we ‘d execute is typing

```
>> kmcdriver
```

Figure 1 shows the results for initial conditions of $x_1(0) = x_2(0) = 0$.

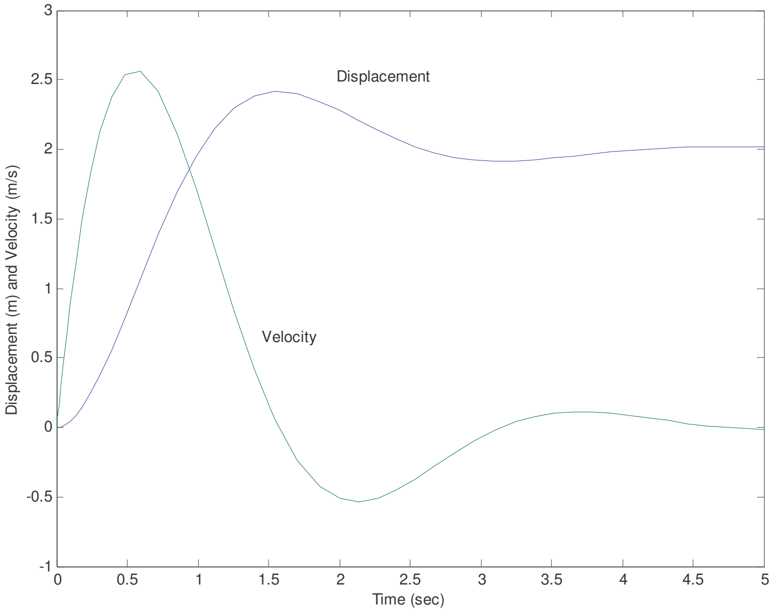

Figure 1 – Linear Spring Mass Damper System

To find the eigenvalues and eigenvectors of the above linear system, one would simply add in the eig( ) MATLAB function call.

Summary of MATLAB Linear Algebra Functions

Command	Description
det(A)	Returns the determinant of the matrix A in symbolic form
eig(A)	Returns the eigenvalues (characteristic roots) of the matrix A in symbolic form
inv(A)	Returns the inverse of the matrix A in symbolic form
poly(A)	Returns the characteristic polynomial of the matrix A in symbolic form

A large number of array functions are provided to aid you in performing operations on matrices some examples of which follow:

`>> A = [0.5 0.5 0.5; 0.5 1.5 1.5; 0.5 1.5 2.5]`	
`>> detA = det(A)`	returns the determinant of A into the scalar named detA
`>> Ainv = inv(A)`	returns the inverse of A into the array named invA
`>> y = eig(A)`	computes the eigenvalues of A into column vector y
`>> [x1,x2] = eig(A)`	returns the the matrices U and D, D is a diagonal matrix with the eigenvalues along the diagonal, U is a full matrix with the eigenvectors as column vectors

Numerical Linear Algebra Using MATLAB

Example 2) Solve the following system of equations using the matrix inverse method

$$\begin{aligned} 2x + 9y &= 5 \\ 3x - 4y &= 7 \end{aligned}$$

The matrix [A] in this case is simply

$$[A] = \begin{bmatrix} 2 & 9 \\ 3 & -4 \end{bmatrix}$$

Its determinant is given by

$$|A| = 2(-4) - 9(3) = -35$$

and its inverse corresponds to

$$[A]^{-1} = \frac{1}{-35}\begin{bmatrix} -4 & -9 \\ -3 & 2 \end{bmatrix} = \frac{1}{35}\begin{bmatrix} 4 & 9 \\ 3 & -2 \end{bmatrix}$$

Consequently, the solution vector is given by

$$\{x\} = [A]^{-1}\{b\} = \frac{1}{35}\begin{bmatrix} 4 & 9 \\ 3 & -2 \end{bmatrix}\begin{Bmatrix} 5 \\ 7 \end{Bmatrix} = \begin{Bmatrix} 2.3714 \\ 0.0286 \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

The corresponding MATLAB commands would read

```
>> A = [2,9;3,-4];
>> b = {5;7]
>> x = inv(A)*b
x =
    2.3714
    0.0286
```

Example 3) Characteristic roots using the eig( ) MATLAB function. Consider the system of ODEs given below

$$\begin{aligned} \dot{x} &= -3x + y \\ \dot{y} &= -x - 7y \end{aligned}$$

We can obtain the characteristic polynomial and roots for a system of linear ODEs working in state-space form by plugging in the following assumed form of the solutions

$$\begin{aligned} x(t) &= C_1 e^{\lambda t} \\ y(t) &= C_2 e^{\lambda t} \end{aligned}$$

where C_1 and C_2 are constants, affords the linear system of equations

$$\begin{aligned} \lambda C_1 e^{\lambda t} &= -3C_1 e^{\lambda t} + C_2 e^{\lambda t} \\ \lambda C_2 e^{\lambda t} &= -C_1 e^{\lambda t} - 7C_2 e^{\lambda t} \end{aligned}$$

Collecting like terms and dividing each side by the exponential term yields

$$\begin{aligned} (\lambda + 3)C_1 - C_2 &= 0 \\ C_1 + (\lambda + 7)C_2 &= 0 \end{aligned}$$

Recall, a solution to the above system exists if and only if the determinant is non-zero, this requires that

$$\begin{vmatrix} \lambda+3 & -1 \\ 1 & \lambda+7 \end{vmatrix} = \lambda^2 + 10\lambda + 22 = 0$$

The above is recognizable as the characteristic polynomial (equation) and its roots are given by

$$\lambda_1 = -6.7321$$
$$\lambda_2 = -3.2679$$

MATLAB provides the eig( ) function to compute the characteristic roots of the characteristic polynomial when the system model is given in the state-variable form. To find the eigenvalues of the above problem, the MATLAB dialog would simply be given as follows:

```
>> A = [-3,1;-1,-7];
>> r = eig(A)

r = [-6.7321, -3.2679]
```

To find the time constants of the system, which are the negative reciprocals of the real parts of the roots, i.e.

$$\tau_1 = \frac{-1}{\mathrm{Re}(\lambda_1)}$$
$$\tau_2 = \frac{-1}{\mathrm{Re}(\lambda_2)}$$

one would simply key in the MATLAB command

```
>> tau = -1./real(r)

tau = [0.1485, 0.3060]
```

Symbolic Linear Algebra

Creating a symbolic matrix can be accomplished in many ways as shown below

```
>> A = sym([3,5;2,7]);
>> B = [3, 5;2,7];
>> C = sym(B)
```

```
>> D = subs(A, [3,5 ; 2, 7]);
```

The first method is the most direct. Use the second method when you want to keep a numeric version of the matrix. The matrices A and C are symbolic and identical. The matrices B and D look like A and C but are numeric of class double.

Example 4) You can also create a symbolic matrix comprised of functions. Consider the relationship between the coordinates *(x,y)* of a coordinate system through an angle α relative to the *(X,Y)* coordinate system

$$X = x\cos\alpha + y\sin\alpha$$
$$Y = y\cos\alpha - x\sin\alpha$$

These equations can be expressed in matrix form as

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = [R(\alpha)] \begin{Bmatrix} x \\ y \end{Bmatrix}$$

where the rotation matrix [*R*] has been defined as

$$[R] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

The symbolic matrix [*R*] can be defined in MATLAB as follows:

```
>> syms a
>> R = [cos(a), sin(a); -sin(a), cos(a)]
R = [cos(a) , sin(a) ]
    [-sin(a), cos(a)]
```

If the coordinate system is rotated twice by the same angle to produce a third coordinate system (*x' , y')*, say, the result is the same as a single rotation with twice the angle.

Thus,

$$\begin{Bmatrix} x' \\ y' \end{Bmatrix} = [R] \begin{Bmatrix} X \\ Y \end{Bmatrix} = [R][R] \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Hence, [*R*(α)][*R*(α)] should be equivalent to [*R*(2α)]. Continuing the MATLAB dialog started above,

```
>> Q = R*R
Q =
```

```
      [cos(a)^2-sin(2)^2, 2*cos(a)*sin(a)]
      [-2*cos(a)*sin(a), cos(a)^2-sin(a)^2]
>> Q = simple(a)
Q =

      [cos(2*a), sin(2*a)]
      [-sin(2*a), cos(2*a)]
```

The matrix $[Q] = [R(2\alpha)]$ as expected. To evaluate a symbolic matrix numerically, use the subs ( ) and double( ) functions, i.e. for a rotation of $\alpha = \pi/4$ radians (45 °),

```
>> Q = subs(Q, a, pi/4)
```

Example 5) Characteristic polynomial and roots of a symbolic matrix. Consider the state-space form of a system of first order ODEs given by

$$\{\dot{x}\} = [A]\{x\} + \{b\} f(t)$$

where

$$[A] = \begin{bmatrix} 0 & 1 \\ -k & -2 \end{bmatrix}, \ \vec{x} = \begin{Bmatrix} x \\ y \end{Bmatrix}, \ \vec{b} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

From our knowledge of eigenvalue theory, the equation $|\lambda[I] - [A]| = 0$ is the characteristic equation of the system model. We can use the poly( ) function in MATLAB to build the characteristic polynomial as follows

```
>> syms k
>> A = [0. 1; -k, -2];
>> poly(A)

ans = x^2 + 2*x + k
>> solve(ans)
ans =
      [-1+(1-k)^(1/2) ]
      [-1-(1-k)^(1/2) ]
```

Thus, the roots are given as $\lambda_{1,2} = -1 \pm \sqrt{1-k}$. Using the eig( ) function bypasses the determination of the characteristic polynomial, i.e.

```
>> syms k
>> A = [0. 1; -k, -2];
```

```
>> eig(A)

ans =
     [-1+(1-k)^(1/2) ]
     [-1-(1-k)^(1/2) ]
```

Finally, one can use the inv( ) and det( ) functions to invert and find the determinant of a matrix symbolically. For example, continuing with the above matrix

```
>> inv(A)
ans =
     [-2/k,-1/k]
     [0,1]
>> A*ans

ans= [1,0]
     [0,1]

>> det(A)
ans =
     k
```

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Using XNUMBERS Add-In Library With Excel To Perform Linear Alegbra & Eigenvalue Analysis

http://www.csupomona.edu/~kranderson1/ME232/me232_index.html

XNUMBERS Numeric Calculus in EXCEL Add-Ins Homepage