DESIGN OF FIR DIFFERENTIATOR

If a continuous time signal x(t) is differentiated in the time domain, we have

$$\frac{dx(t)}{dt}.$$

The Laplace transform of x'(t) is given by

$$L\left[\frac{dx(t)}{dt}\right] = sX(s)$$

where initial condition is ignored.

If x(t) is applied to a differentiator, the output is given by y(t) = dx(t)/dt = x'(t). The Fourier transform of the output y(t) (= x'(t)) is given by

$$Y(\omega) = F\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega)$$

The ratio of Y(ω) over X(ω) is defined as the transfer function H(ω) of the differentiator. Thus,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = j\omega$$

Similarly, the transfer function of a discrete time differentiator is defined as

$$H_d(\theta) = j\theta, \qquad -\pi \le \theta \le \pi$$

Notice that a differentiator introduce a phase shift of π/2, and that it emphasizes high frequency components and de-emphasizes low frequency components. The impulse response of a differentiator is obtained by taking the inverse discrete time Fourier transform of $H_d(\theta)$. Thus,

$$h_d(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\theta)e^{j\theta k}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\theta e^{j\theta k}d\theta$$

Using

$$\int xe^{ax}dx = \frac{1}{a^2}e^{ax}(ax-1)$$

we have

$$h_d(k) = \frac{j}{2\pi}\left[\frac{1}{(jk)^2}e^{jk\theta}(jk\theta-1)\right]_{-\pi}^{\pi} = \frac{j}{2\pi}\left[\frac{e^{jk\pi}(jk\pi-1)}{-k^2} - \frac{e^{jk(-\pi)}(-jk\pi-1)}{-k^2}\right]$$

$$= \frac{j}{2\pi}\left[\frac{jk\pi e^{jk\pi} + jk\pi e^{-jk\pi}}{-k^2} + \frac{-e^{jk\pi} + e^{-jk\pi}}{-k^2}\right] = \frac{1}{2\pi}\left[\frac{2\pi\cos(k\pi)}{k} + \frac{\sin(k\pi)}{k^2}\right] = \frac{\cos(k\pi)}{k}$$

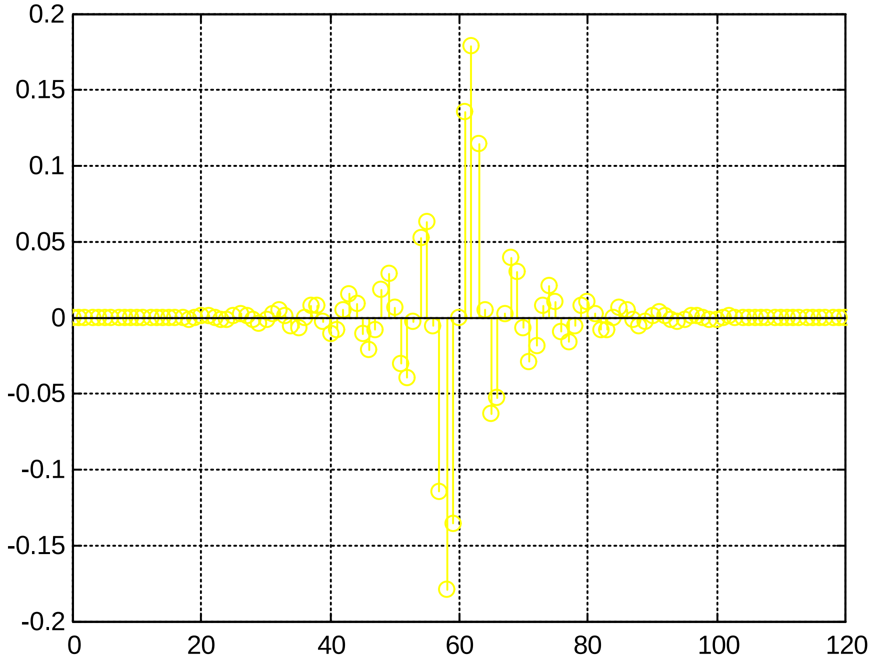
for k ≠ 0. For k = 0, we have

$$h_d(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\theta e^{j\theta 0}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} j\theta d\theta = 0$$

Notice that $h_d(k)$ is an odd function of k.

The differentiator can be designed using Parks-McCllelan algorithm. The following MATLAB file illustrates differentiator design using Parks-McCllelan algorithm. Notice that we can design piecewise linear filter using Parks-McCllelan algorithm.

```
%File Name: diff1.m
%Written by Dr. James S. Kang, ECE Department, Cal Poly Pomona.
%Design of a differentiator using Parks-McCllelan algorithm.
[x,fs,bits]=wavread('chimes.wav');
%fs=sampling rate
disp('sampling rate = ')
disp(fs)
soundsc(x)   %or sound(x)
pause(2)
fs2=fs/2;
N=121;
f=[0,0.3,0.4,1];
m=[0,1,0,0];
h=remez(N-1,f,m,'differentiator');
a=[1];
[H,w]=freqz(h,a,100);
n=1:N;
stem(n-1,h);
grid
figure
plot(w/pi,abs(H));
grid
y=filter(h,a,x);
soundsc(y)   %or sound(y)
```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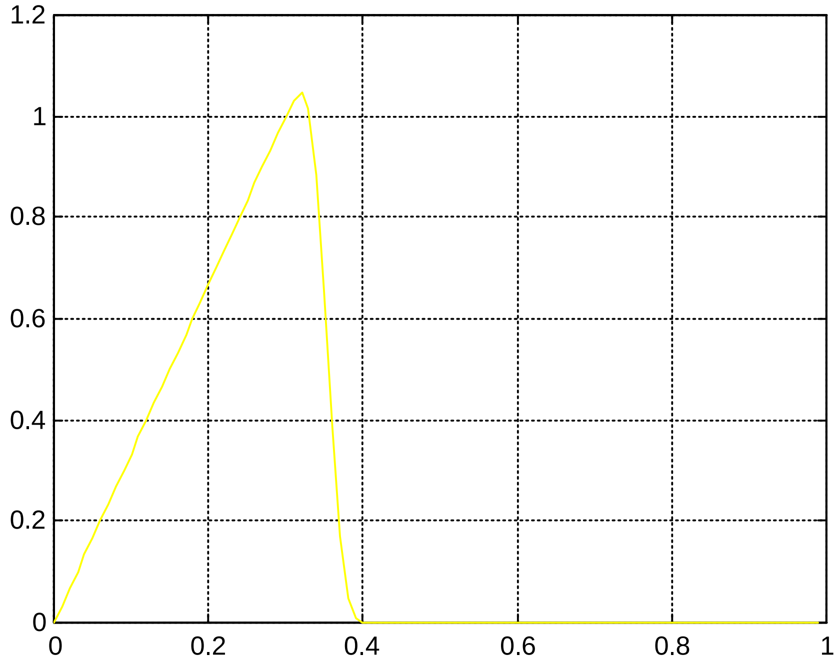
0
-0.05
-0.1
-0.15
-0.2
0
20
40
60
80
100
120


1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

DESIGN OF HILBERT TRANSFORMER

A Hilbert transformer is a device that shifts the phase of the input signal by $-\pi/2$ for all frequency components. In other words, a Hilbert transformer is a phase shifter with phase shift of $-\pi/2$. The transfer function of the Hilbert transformer is given by

$$H(\omega) = \begin{cases} -j, & \omega > 0 \\ 0, & \omega = 0 \\ j, & \omega < 0 \end{cases}$$

or

$H(\omega) = -j\,\mathrm{sgn}(\omega)$.

The Hilbert transform $\hat{x}(t)$ of a continuous time signal x(t) with Fourier transform $X(\omega)$ can be obtained in the frequency domain by

$\hat{x}(t) = F^{-1}\left[-j\,\mathrm{sgn}(\omega)X(\omega)\right]$.

In the time domain, the Hilbert transform of x(t) is obtained by the convolution of x(t) and $1/(\pi t)$, that is,

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\,d\tau$$

In the discrete time domain, the transfer function of the Hilbert transformer is given by

$$H_d(\theta) = \begin{cases} -j, & \theta > 0 \\ 0, & \theta = 0 \\ j, & \theta < 0 \end{cases}$$

The impulse response of the Hilbert transformer is obtained by taking the inverse discrete time Fourier transform of $H(\theta)$. Thus,

$$h_d(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\theta)e^{j\theta k}d\theta = \frac{1}{2\pi}\left[\int_{-\pi}^{0} je^{j\theta k}d\theta - \int_{0}^{\pi} je^{j\theta k}d\theta\right]$$

$$= \frac{1}{2\pi}\left[j\frac{e^{j\theta k}\Big|_{-\pi}^{0}}{jk} - j\frac{e^{j\theta k}\Big|_{0}^{\pi}}{jk}\right] = \frac{1}{2\pi}\left[\frac{1-e^{-j\pi k}}{k} - \frac{e^{j\pi k}-1}{k}\right] = \frac{1}{2\pi}\left[\frac{2-\left(e^{-j\pi k}+e^{j\pi k}\right)}{k}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2-2\cos(\pi k)}{k}\right] = \frac{1-\cos(\pi k)}{\pi k} = \frac{2\sin^2\left(\frac{\pi k}{2}\right)}{\pi k}, \quad k \neq 0$$

When k = 0,

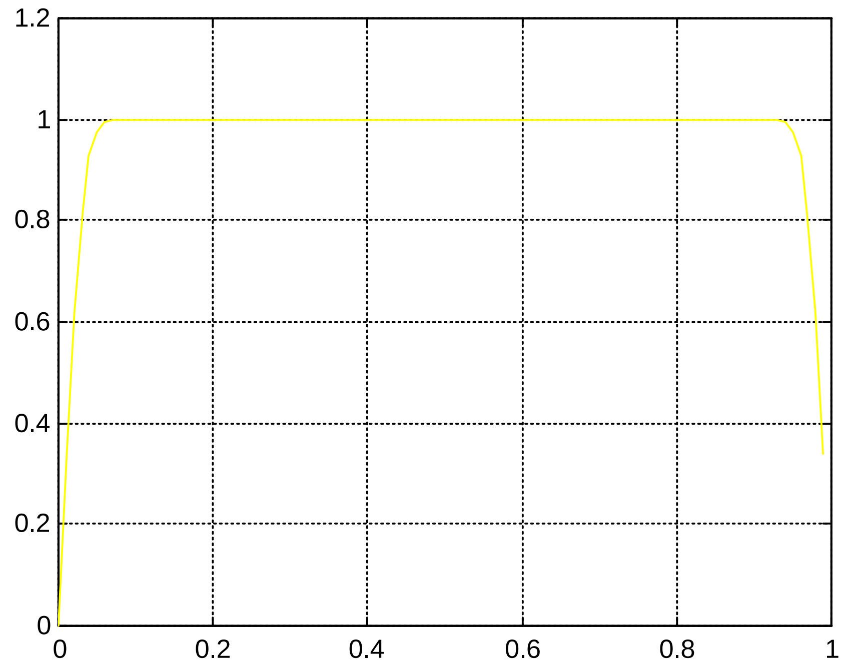
$$h_d(0) = \frac{1}{2\pi}\left[\int_{-\pi}^{0} je^{j\theta 0}d\theta - \int_{0}^{\pi} je^{j\theta 0}d\theta\right] = \frac{1}{2\pi}\left[\int_{-\pi}^{0} jd\theta - \int_{0}^{\pi} jd\theta\right] = 0$$

Notice that $h_d(k)$ is an odd function of k.

The Hilbert transformer can be designed using Parks-McCllelan algorithm. The following MATLAB file illustrates Hilbert transformer design using Parks-McCllelan algorithm.

```
%File Name: Hilbert1.m
%Written by Dr. James S. Kang, ECE Department, Cal Poly Pomona.
%Design of a Hilbert transformer using Parks-McCllelan algorithm.
[x,fs,bits]=wavread('chimes.wav');
%fs=sampling rate
disp('sampling rate = ')
disp(fs)
soundsc(x)   %or sound(x)
pause(2)
fs2=fs/2;
N=121;
f=[0.1, 0.9];
m=[1, 1];
h=remez(N-1,f,m,'Hilbert');
a=[1];
[H,w]=freqz(h,a,100);
n=1:N;
stem(n-1,h);
grid
figure
plot(w/pi,abs(H));
grid
y=filter(h,a,x);
soundsc(y)   %or sound(y)
```

Notice that the magnitude response is flat for the Hilbert transformer. Hilbert transformer is an allpass filter that changes the phase only.



1.2
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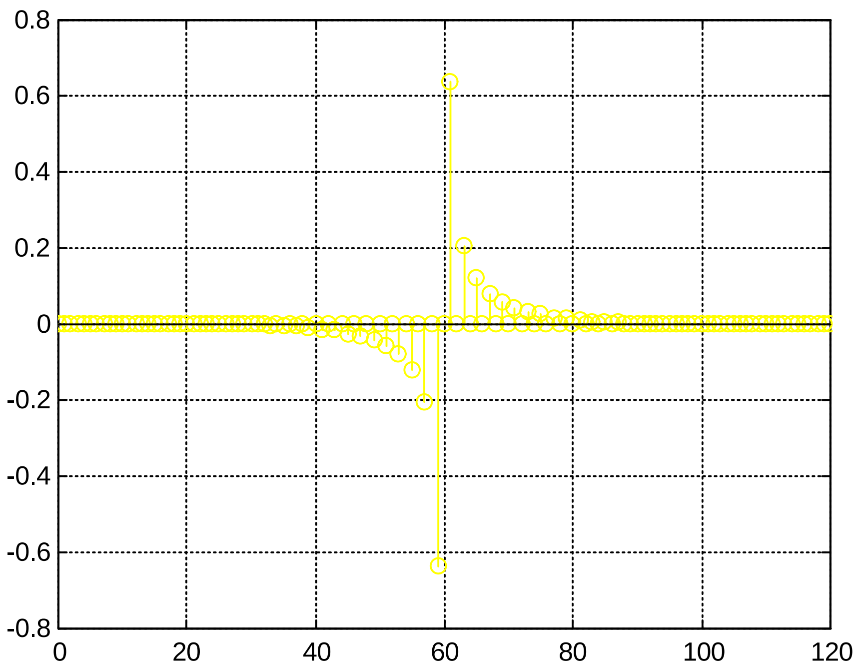
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1



0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
20
40
60
80
100
120

REFERENCES

[1] Andreas Antoniou, *Digital Filters: Analysis, Design, and Applications*, Second Edition, McGraw-Hill, New York, 1993.

[2] T. W. Parks and C. S. Burrus, *Digital Filter Design*, Wiley, New York, 1987.

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[5] L. R. Rabiner, J. H. McClellan, and T. W. Parks, "FIR Digital Filter Design Techniques Using Weighted Chebyshev Approximation," *Proc. IEEE*, vol. 63, pp. 595-610, April 1975.

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