

**The n-Dimensional Deviation Theorem and
the Magnitudes of the Defining Functions of
the n-Dimensional means**

Frank Glaser

Mathematics

The linear ordering theorem for weighted means of functions relative to weight functions (**Glaser, 2001**) has several interesting implications similar to those of the one-dimensional **Cashwell-Everett** mean. (**Cashwell-Everett, 1969**). As an immediate consequence of the linear ordering theorem, we develop in this article the **n**-dimensional deviation theorem and a theorem dealing with inequalities related to the magnitudes of the defining functions of the **n**-dimensional means.

The n-Dimensional Deviation Theorem.

Hypothesis:

- (1) Let $f_2(\vec{v}_1)$ and $h_2(\vec{v}_1)$ be two continuous functions defined on the interval $I_1 = [\alpha, \beta]$ such that $f_1(\vec{v}_1) \leq h_1(\vec{v}_1)$ for all $\vec{v}_1 \in I_1$ where $\vec{v}_1 \in I_1$ when $\vec{v}_1 = \langle v_1 \rangle$ is a vector with one component v_1 .
- (2) Let $f_2(\vec{v}_1)$ and $h_2(\vec{v}_1)$ be two continuous functions defined on the closed region $I_2 = [\alpha, \beta] \times [\inf_1, f_1, \sup_1, h_1]$ such that $f_2(\vec{v}_2) \leq h_2(\vec{v}_2)$ for all $\vec{v}_2 \in I_2$ where $\vec{v}_2 = \langle v_1, v_2 \rangle$ is a vector with two components v_1 and v_2 .
- (3) In general, let $f_j(\vec{v}_j)$ and $h_j(\vec{v}_j)$ be two continuous functions defined on the closed j -dimensional region $I_j = \left[\begin{matrix} \inf f_1, & \sup h_1 \\ i_1 & l_1 \end{matrix} \right] \times \dots \times \left[\begin{matrix} \inf f_{j-1}, & \sup h_{j-1} \\ l_{j-1} & l_{j-1} \end{matrix} \right]$ such that $f_j(\vec{v}_j) \leq h_j(\vec{v}_j)$ for all $\vec{v}_j \in I_j$ where $\vec{v}_j = \langle v_j, \dots, v_j \rangle$ is a vector with j components for $j = 1, \dots, m$.
- (4) Let $x_k = x_k(\vec{v}_m)$ be n non-constant continuous function defined on the closed m -dimensional region $I_m = [\alpha, \beta] \times \left[\begin{matrix} \inf f_1, & \sup h_1 \\ l_1 & l_1 \end{matrix} \right] \times \dots \times \left[\begin{matrix} \inf f_{m-1}, & \sup h_{m-1} \\ l_{m-1} & l_{m-1} \end{matrix} \right]$ where they have only a finite number of maxima and minima. Also let $\inf_{I_m} x_k(\vec{v}_m) = x_{ki}$ and $\sup_{I_m} x_k(\vec{v}_m) = x_{ks}$.
- (5) Let $W_k(\xi_k, \vec{v}_m)$, for fixed k , be a weight function defined and continuous on the Cartesian product $[x_{ki}, x_{ks}] \times I_m$ where it is positive almost everywhere.
- (6) Let $g_k(\xi_k)$ be a continuous, strictly increasing and positive almost everywhere function on the closed interval $[x_{ki}, x_{ks}]$.
- (7) On the interval $[x_{ki}, x_{ks}]$ define the four following functions:

$$A_k(x_k) = \int_{V_m} \int_{x_k(\vec{v}_m)}^{x_k} \left\{ g_k[x_k(\vec{v}_m)] \right\}^{-1} g_k(\xi_k) W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$$

$$B_k(x_k) = \int_{V_m} \int_{x_k(\vec{v}_m)}^{x_k} [g_k(\xi_k)]^{-1} g_k(\xi_k) W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$$

$$C_k(x_k) = \int_{V_m} \int_{x_k(\vec{v}_m)}^{x_k} g_k(\xi_k) W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$$

$$D_k(x_k) = \int_{V_m} \int_{x_k(\vec{v}_m)}^{x_k} g_k[x_k(\vec{v}_m)] W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$$

- (8) Let a_k, b_k, c_k, d_k be the unique real numbers such that $A_k(a_k) = 0, B_k(b_k) = 0, C_k(c_k) = 0, D_k(d_k) = 0$.

Conclusion: Then the following inequalities hold.

$$(I) \quad \sum_{k=1}^n (b_k - a_k)^2 < \sum_{k=1}^n (c_k - a_k)^2 < \sum_{k=1}^n (d_k - a_k)^2$$

$$(II) \quad \sum_{k=1}^n (c_k - b_k)^2 < \sum_{k=1}^n (d_k - b_k)^2$$

This theorem is a simple corollary of the linear ordering theorem. It gives us inequalities relating the distances in Euclidean n-space between the means generated by the defining functions A, B, C, and D. We next obtain an estimate of the magnitude of the distance between the means of the defining functions **B** and **C**.

The magnitude of the Distance between the means generated by $\mathbf{B}(x_1, \dots, x_n)$ and $\mathbf{C}(x_1, \dots, x_n)$.

Theorem 1. Under the same hypotheses as in the n-dimensional deviation theorem the coordinates b_k and c_k of the means generated by $B_k(x_k)$ and $C_k(x_k)$ respectively satisfy the following inequalities

$$0 < -C_k(b_k) < -D_k(b_k) < [g_k(x_{ks}) - g_k(x_{ki})] \min[-B_k(x_{ki}), B_k(x_{ks})]$$

$$0 < B_k(C_k) < A_k(C_k) < \left[\frac{1}{g_k(x_{ki})} - \frac{1}{g_k(x_{ks})} \right] \min[-C_k(x_{ki}), C_k(x_{ks})]$$

Proof: Let J_{b_k} be the union of all simple open subregions of V_m such that $x_k(\vec{v}_m) < b_k$ for all $\vec{v}_m \in J_{b_k}$ and let S_{b_k} be the union of all simple open subregions of V_m such that $x_k(\vec{v}_m) > b_k$ for all $\vec{v}_m \in S_{b_k}$. Then we have

$$C(b_k) = \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} g_k(\xi_k) W_k(\xi_k, \vec{v}_m) d\xi_k dV_m + \int_{S_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} g_k(\xi_k) W_k(\xi_k, \vec{v}_m) d\xi_k dV_m \quad (1)$$

Since $g_k(\xi_k) W_k(\xi_k, \vec{v}_m)$ is non-negative, positive almost everywhere and continuous, we have the following implications. If $\vec{v}_m \in J_{b_k}$ then $\xi_k > x_k(\vec{v}_m)$ and since g_k is monotone increasing we have $g_k(\xi_k) > g_k[x_k(\vec{v}_m)]$ if $\vec{v}_m \in S_{b_k}$ then $\xi_k < x_k(\vec{v}_m)$ and $g_k(\xi_k) < g_k[x_k(\vec{v}_m)]$. This implies that (1) becomes

$$C_k(b_k) > \int_{V_m} \int_{x_k(\vec{v}_m)}^{b_k} g_k[x_k(\vec{v}_m)] W_k(\xi_k, \vec{v}_m) d\xi_k dV_m = D_k(b_k) \quad (2)$$

If $\vec{v}_m \in J_{b_k}$ then $x_{ki} < x_k(\vec{v}_m)$, hence $g_k[x_k(\vec{v}_m)] > g_k(x_{ki})$. If $\vec{v}_m \in S_{b_k}$ then $x_k(\vec{v}_m) < x_{ks}$ and $g_k[x_k(\vec{v}_m)] < g_k(x_{ks})$. This implies that (2) becomes

$$C_k(b_k) > D_k(b_k) > g_k(x_{ki}) \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m + g_k(x_{ks}) \int_{S_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m \quad (3)$$

Now let $\Delta g_k = g_k(x_{ks}) - g_k(x_{ki})$. Since g_k is monotone increasing, $\Delta g_k > 0$. By adding and subtracting

$g_k(x_{ks}) \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$ to the right-hand side of inequality (3) and using the fact that $B_k(b_k) = 0$, we

obtain

$$C_k(b_k) > D_k(b_k) > \Delta g_k(x_{ki}) \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m \quad (4)$$

Now, if $\vec{v}_m \in J_{b_k}$ then $x_k(\vec{v}_m) < b_k < x_{ks}$ and it follows that

$$\Delta g_k(x_{ki}) \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m < \Delta g_k(x_{ki}) \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{x_{ks}} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$$

Furthermore, if $\vec{v}_m \in S_{b_k}$ then $b_k < x_k(\vec{v}_m) < x_{ks}$ and since $\Delta g_k \int_{S_{b_k}} \int_{x_k(\vec{v}_m)}^{x_{ks}} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m > 0$ we now find that

$\Delta g_k(x_{ki}) \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m < \Delta g_k \int_{J_{b_k}} \int_{x_k(\vec{v}_m)}^{b_k} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m + \int_{S_{b_k}} \int_{x_k(\vec{v}_m)}^{x_{ks}} W_k(\xi_k, \vec{v}_m) d\xi_k dV_m$ and hence,

$$\Delta g_k \int_{J_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) dV_m < \Delta g_k B_k(x_{ks}).$$

This implies that inequality (4) becomes

$$C_k(b_k) D_k(b_k) > \Delta g_k \int_{J_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m > -\Delta g_k B_k(x_{ks}) \quad (5)$$

Since $g_k(x_{ks}) = g_k(x_{ki}) + \Delta g_k$, inequality (3) now becomes

$$C(b_k) > D_k(b_k) > g_k(x_{ki}) B_k(b_k) + \Delta g_k \int_{S_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m \text{ and since } B_k(b_k) = 0 \text{ we obtain}$$

$$C_k(b_k) > D_k(b_k) > \Delta g_k \int_{S_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m \quad (6)$$

Now if $\bar{v}_m \in S_{b_k}$ then $x_{ki} < b_k < x_k(\bar{v}_m)$ and since $W_k(\xi_k, \bar{v}_m) > 0$ almost everywhere and $\Delta g_k > 0$, we have

$$\Delta g_k \int_{S_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m > \Delta g_k \int_{S_{b_k} x_k(\bar{v}_m)}^{x_{ki}} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m.$$

Also if $\bar{v}_m \in J_{b_k}$ then $x_{ki} < x_k(\bar{v}_m) < b_k$ and we have $\Delta g_k \int_{J_{b_k} x_k(\bar{v}_m)}^{x_{ki}} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m < 0$ which implies that

$$\Delta g_k \int_{S_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m > \Delta g_k \int_{V_m x_k(\bar{v}_m)}^{x_{ki}} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m$$

or

$$\Delta g_k \int_{S_{b_k} x_k(\bar{v}_m)}^{b_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m > \Delta g_k B_k(x_{ki}) \quad (7)$$

Inequalities (6) and (7) now imply that

$$C_k(b_k) > D_k(b_k) < \Delta g_k B_k(x_{ki}) \quad (8)$$

Since $\Delta g_k = g_k(x_{ks}) - g_k(x_{ki})$, inequalities (5) and (8) can be rewritten as

$$\begin{cases} -C_k(b_k) < -D_k(b_k) < [g_k(x_{ks}) - g_k(x_{ki})][B_k(x_{ks})] \\ -C_k(b_k) < -D_k(b_k) < [g_k(x_{ks}) - g_k(x_{ki})][-B_k(x_{ki})] \end{cases}$$

or simply as

$$-C_k(b_k) < -D_k(b_k) < [g_k(x_{ks}) - g_k(x_{ki})] \min \{-B_k(x_{ki}), B_k(x_{ks})\}$$

This proves the first part of the theorem.

To prove the second part let J_{c_k} be the union of all simple open subregions in V_m such that $x_k(\bar{v}_m) < c_k$ for all $\bar{v}_m \in J_{c_k}$ and S_{c_k} the union of all simple subregions in V_m such that $x_k(\bar{v}_m) > c_k$ for all $\bar{v}_m \in S_{c_k}$ we then have

$$\begin{aligned} B_k(c_k) &= \int_{J_{c_k} x_k(\bar{v}_m)}^{c_k} W(\xi_k, \bar{v}_m) d\xi_k dV_m + \int_{S_{c_k} x_k(\bar{v}_m)}^{c_k} W_k(\xi_k, \bar{v}_m) d\xi_k dV_m \\ &< \int_{J_{c_k} x_k(\bar{v}_m)} \{g_k[x_k(\bar{v}_m)]\}^{-1} g_k(\xi_k) W_k(\xi_k, \bar{v}_m) d\xi_k dV_m + \int_{S_{c_k} x_k(\bar{v}_m)} \{g_k[x_k(\bar{v}_m)]\}^{-1} g_k(\xi_k) W_k(\xi_k, \bar{v}_m) d\xi_k dV_m \end{aligned}$$

Therefore

$$B_k(c_k) < \int_{V_m x_k(\bar{v}_m)}^{c_k} \{g_k[x_k(\bar{v}_m)]\}^{-1} g_k(\xi_k) W_k(\xi_k, \bar{v}_m) d\xi_k dV_m$$

or

$$B_k(c_k) < A_k(c_k) \quad (9)$$

Similarly, we can also show that $B_k(c_k) < A_k(c_k)$

$$< [g_k(x_{ki})]^{-1} \int_{J_{c_k} x_k(\bar{v}_m)}^{c_k} g_k(\xi_k) W_k(\xi_k, \bar{v}_m) d\xi_k dV_m + [g_k(x_{ks})]^{-1} \int_{S_{c_k} x_k(\bar{v}_m)}^{c_k} g_k W_k(\xi_k, \bar{v}_m) d\xi_k dV_m$$

where we add and subtract the term $[g_k(x_{ki})]^{-1} \int_{s_{c_k}}^{c_k} \int_{x_k(\bar{v}_m)}^{c_k} g_k(\xi_k) W_k(\xi_k, \bar{v}_k) d\xi_k dV_m$ in order to obtain the inequality $B_k(c_k) <$

$$A_k(c_k) < [g_k(x_{ki})]^{-1} C_k(c_k) - \{ [g_k(x_{ks})]^{-1} - [g_k(x_{ki})]^{-1} \} \int_{s_{c_k}}^{x_k(\bar{v}_m)} \int_{c_k}^{c_k} g_k(\xi_k) W_k(\xi_k, \bar{v}_k) d\xi_k dV_m$$

$$\text{and since } c_k(c_k) = 0, \text{ we find } B_k(c_k) < A_k(c_k) < \{ [g_k(x_{ki})]^{-1} - [g_k(x_{ks})]^{-1} \} \int_{s_{c_k}}^{x_k(\bar{v}_m)} \int_{c_k}^{c_k} g_k(\xi_k) W_k(\xi_k, \bar{v}_k) d\xi_k dV_m$$

$$\text{This in turn leads to the inequalities } B_k(c_k) < A_k(c_k) < \{ [g_k(x_{ki})]^{-1} - [g_k(x_{ks})]^{-1} \} \int_{s_{c_k}}^{x_k(\bar{v}_m)} \int_{c_k}^{c_k} g_k(\xi_k) W_k(\xi_k, \bar{v}_k) d\xi_k dV_m$$

$$< \{ [g_k(x_{ki})]^{-1} - [g_k(x_{ks})]^{-1} \} \left\{ \int_{s_{c_k}}^{c_k} \int_{x_k(\bar{v}_m)}^{c_k} g_k(x_k) W_k(\xi_k, \bar{v}_k) d\xi_k dV_m + \int_{s_{c_k}}^{x_k(\bar{v}_m)} \int_{c_k}^{c_k} g_k(\xi_k) W_k(\xi_k, \bar{v}_k) d\xi_k dV_m \right\} \text{ and therefore,}$$

$$B_k(c_k) < A_k(c_k) < \left[\frac{1}{g_k(x_{ki})} - \frac{1}{g_k(x_{ks})} \right] C_k(x_{ks}) \quad (10)$$

Similarly, we can also show that

$$B_k(c_k) < A_k(c_k) < \left[\frac{1}{g_k(x_{ki})} - \frac{1}{g_k(x_{ks})} \right] [C_k(x_{ks})]. \quad (11)$$

Inequalities (10 and (11) imply that $B_k(c_k) < A_k(c_k) < \left[\frac{1}{g_k(x_{ki})} - \frac{1}{g_k(x_{ks})} \right] \min\{C_k(x_{ki}), C_k(x_{ks})\}$ which proves the second part of the theorem.

An Example Illustrating the Theorem:

In the example illustrating the linear ordering theorem (Glaser, 2001), We had the eight integrals:

$$A_k(X_k) = \int_0^{1/v_k} \int_0^{X_k} \int_0^{v_k} v_k^{-1} \xi_k (v_1 + v_2) d\xi_k dv_2 dv_1 = \frac{13}{48} x_1^2 - \frac{11}{240}$$

$$B_k(X_k) = \int_0^{1/v_k} \int_0^{X_k} \int_0^{v_k} \xi_k^{-1} \xi_k (v_1 + v_2) d\xi_k dv_2 dv_1 = \frac{3}{20} x_1 - \frac{11}{120}$$

$$C_k(X_k) = \int_0^{1/v_k} \int_0^{X_k} \int_0^{v_k} \xi_k (v_1 + v_2) d\xi_k dv_2 dv_1 = \frac{3}{40} x_1^2 - \frac{13}{420}$$

$$D_k(X_k) = \int_0^{1/v_k} \int_0^{X_k} \int_0^{v_k} v_k (v_1 + v_2) d\xi_k dv_2 dv_1 = \frac{11}{20} x_1 - \frac{13}{210}$$

The x_k values that make these integrals zero were $a_k \cong 0.41$, $b_k \cong 0.61$, $c_k \cong 0.64$, $d_k \cong 0.67$ where $k = 1, 2$. We also had $x_{ki} = 0$, $x_{ks} = 1$ and $g_k(k) = k$.

The inequality: $0 < C_k(c_k) < -D_k(d_k) < [g_k(x_{ks}) - g_k(x_{ki})] \min[-B_k(x_{ki}), B_k(x_{ks})]$ is verified, since for $k = 1, 2$, we have $0 < -(-0.003045) < -(-0.005988) < 0.058333$.

The second inequality: $0 < B_k(c_k) < A_k(c_k) < \left[\frac{1}{g_k(x_{ki})} - \frac{1}{g_k(x_{ks})} \right] \min[-C_k(x_{ki}), C_k(x_{ks})]$ is also verified, since for

$k = 1, 2$, we have $\lim_{\xi_k \rightarrow x_{ki}} \frac{1}{g_k(\xi_k)} = \infty$ $\lim_{\xi_k \rightarrow 0^+} \frac{1}{\xi_k} = \infty$ and therefore, we obtain: $0 < 0.004 < 0.06495 < \infty$.

CONCLUSION:

In this article, we have established a corollary of the linear ordering theorem, the N-dimensional deviation theorem, followed by two fundamental inequalities governing the magnitudes of the defining functions $B_k(x_k)$ and $C_k(x_k)$ of means b_k and c_k respectively. This will be followed by two theorems dealing with the magnitudes of the distance between the means.

REFERENCES:

- Cashwell, E. D. and Everett, J. C. (1969), The mean of a function $X(V)$ relative to a weight function $W(\xi, v)$. American Mathematical Monthly, 77m 603-609.
- Glaser, F. (2001), The Journal of Interdisciplinary Studies, Volume 13, Fall 2000.