A THEOREM FOR THE DETERMINATION OF VERTICAL AND HORIZONTAL TANGENTS TO THE HYPERPOLAR IMAGE OF A CIRCLE

Frank Glaser **Mathematics**

The aim of this article is to complete the basic theory of the hyperpolar images of Cartesian circles as developed in a previous article (Glaser, 1992). These images consist of a certain number of loops that depends on the radius of the circle that has been mapped into the hyperpolar plane. In order to have a complete theory of these curves and to be able to draw their graphs with precision, this theorem is of great importance. Two examples showing its application will also be considered.

The Vertical-Horizontal Tangent Theorem

It has been shown that the general solution of the differential equation

$$\left[2u + \tan^{-1}\left(\frac{v}{u}\right) + v\ln\left(u^2 + v^2\right)\right]\frac{dv}{du} + u\ln\left(u^2 + v^2\right) - 2v\tan^{-1}\left(\frac{v}{u}\right) = 0$$

is

$$\left[\tan^{-1}\left(\frac{v}{u}\right)\right]^2 \left[\ln\sqrt{u^2 + v^2}\right]^2 = r^2 \qquad (1)$$

which is the equation of a one-parameter family of curves whose basic properties have been studied in a previous article (Glaser, 1992). This family of curves can be parametrized by writing its position vector in the form

$$\vec{r}(x) = \exp\left[\pm\sqrt{r^2 - x^2}\right]\left\langle \cos x, \sin x \right\rangle \qquad (2)$$

where $|x| \le r$.

The real number r determines the shape of the graph traced out by $\vec{r}(x)$; if r increases, $\vec{r}(x)$ traces out more and more loops which in turn results in an increase in the number of vertical and horizontal tangents. For $\frac{2n-1}{2}\pi < r \le n\pi$, the determination of all vertical tangents presents no difficulties but, depending on the size of r, there can be either a certain known number of horizontal tangents, or if r is somewhat larger, four more, two local minima and two local maxima, which can coalesce into an inflection point if r is taken somewhat smaller, resulting in this case in only two more horizontal tangents in addition to the known number. For $(n-1)\pi < r \le \frac{2n-1}{2}\pi$, a similar result is obtained for vertical tangents.

We now proceed to state and prove a general theorem which will allow us to predict the number of vertical and horizontal tangents to hyperpolar image of the circle $x^2 + y^2 = r^2$ when the value of r is known and $\frac{2n-1}{2}\pi < r \le n\pi$, $n = 1, 2, \ldots$.

Theorem.

Hypothesis:

(1) Given the hyperpolar circle

$$\vec{r}\,(x) = \exp\left[\,\pm\sqrt{r^2 - x^2}\,\right]\left\langle \cos x, \sin x \right\rangle$$

where $| x | \leq r \leq$ and $n\pi \ \frac{2n-1}{2}\pi < | x | \leq \pi$, $n = 1,2,....$

(2) Let $\overline{x} = x + n\pi$ for $-n\pi < x < 0$

and $\overline{x} = x - n\pi$ for $0 < x < n\pi$

(3) Let $\overline{x}_0 \ \mathrm{E} \left(0, \frac{\pi}{2}\right)$ and $-\overline{x}_0 \ \mathrm{E} \left(-\frac{\pi}{2}, 0\right)$ be solutions of $\csc x = n\pi - \mathrm{E}$ and $\csc x = -(n\pi - \mathrm{E})$ respectively.

(4) Let $\mathrm{E} = n\pi - r$.

(5) Define the functions

$D^+(\overline{x}, \mathrm{E}) = n\pi - \overline{x} - (n\pi - \mathrm{E}) \cos \overline{x}$, $0 < \overline{x} < \frac{\pi}{2}$ and

$D^-(\overline{x}, \mathrm{E}) = -n\pi - \overline{x} + (n\pi - \mathrm{E}) \cos \overline{x}$, $-\frac{\pi}{2} < \overline{x} < 0$.

Conclusion

Part I:

Then the values of x for which there are

(1) vertical tangents to the graph of

$$\vec{r}\,(x) = \exp\left[\sqrt{r^2 - x^2}\,\right]\left\langle \cos x, \sin x \right\rangle$$

are given by

$$v_0^+ = 0, v_k^+ \ \mathrm{E} \left(\frac{2k-1}{2}\pi, k\pi\right), -v_k^+ \ \mathrm{E} \left(-kx, -\frac{2k-1}{2}\pi\right)$$

where $k = 1,2,..., n$ and $\left(\frac{2n-1}{2}\right)\pi < r < \pi n$.

(2) vertical tangents to the graph of

$$\vec{r}\,(x) = \exp\left[\,-\sqrt{r^2 - x^2}\,\right]\left\langle \cos x, \sin x \right\rangle$$

are given by

$$v_0^- = 0,\ v_k^- \in \left(k\pi, \frac{2k+1}{2}\pi\right),\ -v_k^- \in \left(-\frac{2k+1}{2}\pi, -k\pi\right)$$

where k = 1,2,...,n-1 and $\frac{2n-1}{2}\pi < r < n\pi$.

Part II:

Furthermore,

(1) the values of x for which there are horizontal tangents to the graph of

$$\vec{r}\,(x) = \exp\left[\sqrt{r^2 - x^2}\right] < \cos x, \sin x >.$$

are given by

$$h_k^+ \in \left(k\pi, \frac{2k+1}{2}\pi\right),\ -h_k^+ \in \left(-\frac{2k+1}{2}\pi, -k\pi\right)$$

where k = 0,1,2,...,n-1 and $\left(\frac{2n-1}{2}\right)\pi < r < n\pi$.

If r = nπ, then there is a horizontal tangent for $x = h_n^+ = n\pi$ and one for $x = -h_n^+ = -n\pi$ where

$$\vec{r}\,(n\pi) = < -1, 0 > = \vec{r}\,(-n\pi);$$

i. e., these two tangents coincide.

(2) The number of horizontal tangents to the graph of

$$\vec{r}\,(x) = \exp\left[-\sqrt{r^2 - x^2}\right] < \cos x, \sin x >$$

depends on the algebraic signs of $D^+(\bar{x}_0, E)$ and $D^-(-\bar{x}_0, E)$

Case 1. $D^+(\bar{x}_0, E) > 0$ and $D^-(-\bar{x}_0, E) < 0$

Then there are horizontal tangents for x equal to

$$h_k^- \in \left(\frac{2k-1}{2}\pi, k\pi\right),\ -h_k^- \in \left(-k\pi, -\frac{2k-1}{2}\pi\right)$$

where k = 1, 2, ..., n-1 and $\left(\frac{2n-1}{2}\right)\pi < r < n\pi$.

Case 2. $D^+(\bar{x}_0, E) = 0 = D^-(-\bar{x}_0, E)$.

Then, in addition horizontal tangents like those in Case 1, there are two more for x equal to

$$h_n^- = x_0 = n\pi - \bar{x}_0,\ -h_n^- = -x_0 = -n\pi + \bar{x}_0$$

and the graph has inflection points for these values of x. This happens for a special value of E which is denoted by E_0.

Case 3. $D^+(\bar{x}_0, E) < 0$ and $D^-(-\bar{x}_0, E) > 0$.

Then, in addition to horizontal tangents like those in Case 1, there are four more for x equal to

$$h_n^- = a = n\pi - \bar{a},\ -h_n^- = -a = -n\pi + \bar{a}$$

$$h_{n+1}^- = b = n\pi - \bar{b},\ -h_{n+1}^- = -b = -n\pi + \bar{b}$$

where $\bar{a}$ and $\bar{b}$ are two solutions of $D^+(\bar{x}, E) = 0$ such that $0 < \bar{a} < \bar{x}_0 < \bar{b} < \frac{p}{2}$ and $-\bar{a}$ and $-\bar{b}$ are two solutions of $D^-(\bar{x}, E) = 0$ such that $-\frac{\pi}{2} < -\bar{b} < -\bar{x}_0 < \bar{a} < 0$.

In this case, the graph of $\vec{r}(x)$ has (i) local minima for $x = h_n^-, -h_{n+1}^-$ if $n = 2k - 1$

and for $x = -h_n^-, h_{n+1}^-$ if $n = 2k$, $(k = 1,2,...)$ (ii) local maxima for $x = -h_n^-, h_{n+1}^-$

if $n = 2k - 1$ and for $x = h_n^-, -h_{n+1}^-$ if $n = 2k (k = 1,2,...)$.

PROOF: The existence of vertical tangents in Part I of the conclusion, and that of horizontal tangents in Part II (1) above, has already been established in the basic theory of hyperpolar images of circles (Glaser, 1992).

Taking the derivative in equation (2), we find

$$\vec{r}\,'(x) = \exp\left[\pm\sqrt{r^2 - x^2}\right]\left\langle \mp\frac{x\cos x}{\sqrt{r^2 - x^2}} - \sin x, \mp\frac{x\sin x}{\sqrt{r^2 - x^2}} + \cos x\right\rangle;$$

for vertical tangents, the first component of this vector must be zero, and for horizontal tangents, the second component vanishes.

Thus, for $y = \sqrt{r^2 - x^2}$ we have vertical tangents if $\tan x = \frac{-x}{\sqrt{r^2 - x^2}}$ and horizontal tangents if $\tan x = \frac{\sqrt{r^2 - x^2}}{x}$. For $y = -\sqrt{r^2 - x^2}$ we have vertical tangents if $\tan x = \frac{x}{\sqrt{r^2 - x^2}}$ and horizontal tangents if $\tan x = -\frac{\sqrt{r^2 - x^2}}{x}$.

The existence of vertical and horizontal tangents is established by graphing the functions $y_t(x) = \tan x, y_v^+(x) = \frac{-x}{\sqrt{r^2 - x^2}}$ and $y_h^-(x) = -\frac{\sqrt{r^2 - x^2}}{x}$ as in Figure 2.

Figure 1. The graphs of $y_t(x)$, $y_v^+(x)$ and $y_h^+(x)$

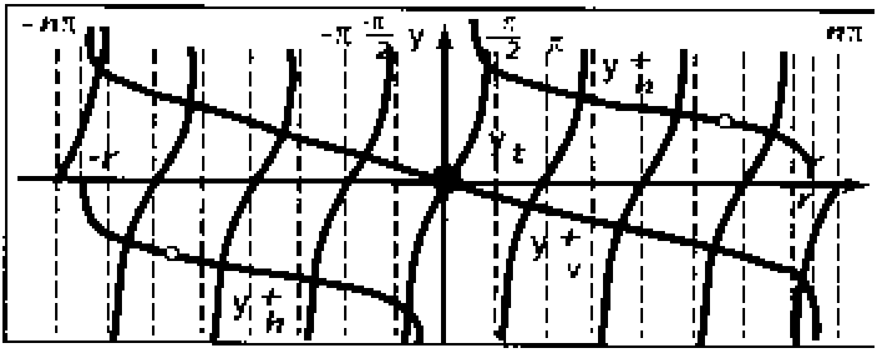
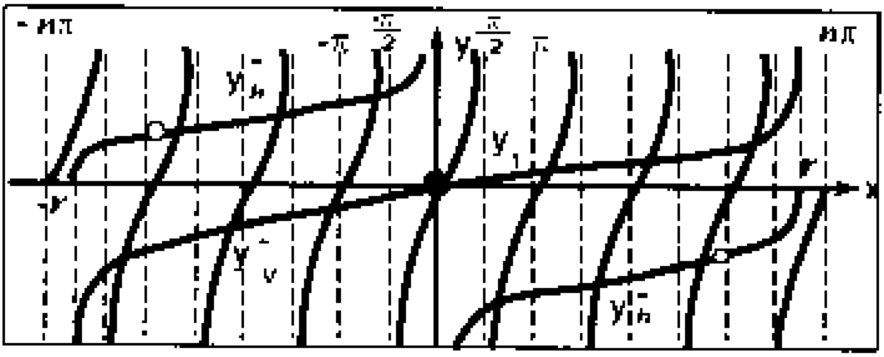


Figure 2. The graphs of $y_t(x)$, $y_v^-(x)$ and $y_h^-(x)$



The x values at which $y_t(x)$ intersections with $y_v^+(x)$ and $y_v^-(x)$ are those x values for which there are vertical tangents and those at which $y_t(x)$ intersects with $y_h^+(x)$ and $y_h^-(x)$ are those for which there are horizontal tangents to the curves given by equation (1) and (2) above.

It can happen that the graph of $y_h^-(x)$ in Figure 2 either intersects or is tangent to that of $y_t(x)$ in the intervals $(-n\pi, -n\pi + E)$ and $(n\pi - E, n\pi)$ for some $E > 0$. We see that for $r = n\pi - E$, there are three cases depending on the magnitude of E.

Case 1. $y_h^-(x)$ does not intersect $y_t(x)$ as shown in Figure 2.

Case 2. $y_h^-(x)$ is tangent to $y_t(x)$ at two points with x coordinates denoted by $-x_0$ and x_0. We denote by E_0 the value of E for which this happens.

Case 3. $y_{\bar{h}}(x)$ intersects $y_t(x)$ at four points where the x coordinates are denoted by -a < -b and b < a.

To prove all this, we introduce two new coordinate systems, one with origin at $x = -n\pi$ and the other with origin at $x = n\pi$, both having their y axes parallel to the y axis in Figure 2. Accordingly, we have two translations of coordinates

$$\bar{x} = x + n\pi \text{ for } -n\pi \le x \le 0$$

$$\text{and } \bar{x} = x - n\pi \text{ for } 0 \le x \le n\pi.$$

We now transform the function $y_t(x)$ and $y_{\bar{h}}(x)$ into $\bar{y}_t(\bar{x}) = y_t(\bar{x} - n\pi)$ and $\bar{y}_{\bar{h}}(\bar{x}) = y_{\bar{h}}(\bar{x} - n\pi)$ for $0 < \bar{x} < \frac{\pi}{2}$ and $|\bar{x} - n\pi| < r$ and also into $\bar{y}_t(\bar{x}) = y_t(\bar{x} + n\pi)$ for $-\frac{\pi}{2} < \bar{x} < 0$ and $|\bar{x} - n\pi| < r$.

For $0 = \bar{x} < \frac{\pi}{2}$ we have $\left[\bar{y}_t(\bar{x})\right]^2 = \tan^2\left(\bar{x} - n\pi\right) = \tan^2 \bar{x}$

$$[\bar{y}_{\bar{h}}(\bar{x})]^2 = \left[\frac{\sqrt{r^2 - (\bar{x} - np)^2}}{(\bar{x} - np)}\right]^2 = \frac{r^2}{(\bar{x} - np)^2} - 1$$

It is easy to show that

$$\left[\bar{y}_t(\bar{x})\right]^2 = \left[\bar{y}_{\bar{h}}(\bar{x})\right]^2 = \frac{\left(\bar{x} - n\pi\right)^2 - r^2\cos^2\left(\bar{x}\right)}{\bar{x} - n\pi}$$

We let $r = n\pi - \mathrm{E}$ and observe that $n\pi - \bar{x} < 0$. Hence,

$$[\bar{y}_t(\bar{x})]^2 - [\bar{y}_{\bar{h}}(\bar{x})]^2 = \frac{\left[(n\pi - \bar{x}) - (n\pi - \mathrm{E}) \cos \bar{x}\right](n\pi - \bar{x}) + (n\pi - \mathrm{E}) \cos \bar{x}}{(n\pi - \bar{x})^2 \cos^2 \bar{x}} \qquad (3)$$

Since $-\frac{\pi}{2} < \bar{x} < 0$, the second factor in the numerator is positive and so is the denominator. Hence, we only need to consider the function

$$D^+(\bar{x}, \mathrm{E}) = n\pi - \bar{x} - (n\pi - \mathrm{E}) \cos \bar{x} \qquad (4)$$

where $0 \le \bar{x} \le \frac{\pi}{2}$ and $\mathrm{E} < 0$, and its partial derivative with respect to $\bar{x}$

$$D^+_{\bar{x}}\left(\bar{x}, \mathrm{E}\right) = -1 + \left(n\pi - \mathrm{E}\right) \sin \bar{x} \qquad (5)$$

This partial derivative is zero for $\bar{x}_0$ such that $\csc \bar{x}_0 = n\pi - \mathrm{E}$ where $0 < \bar{x}_0 < \frac{\pi}{2}$.

For this x_0, we have the second partial derivative

$$D^+_{\bar{x}\bar{x}}(\bar{x}_0, \mathrm{E}) = -(n\pi - \mathrm{E}) \cos \bar{x}_0 < 0$$

and it also follows that

$$\mathrm{E} = n\pi - \csc \bar{x}_0 .$$

We now consider the three cases:

(1) $D^+(\bar{x}_0, \mathrm{E}) > 0$ (2) $D^+(\bar{x}_0, \mathrm{E}) = 0$ (3) $D^+(\bar{x}_0, \mathrm{E}) < 0$.

If $D^+(\bar{x}_0, \mathrm{E}) > 0$ then equation (3) implies that $[\bar{y}_t(\bar{x}_0)]^2 - [\bar{y}_{\bar{h}}(\bar{x}_0)]^2 > 0$ and the graphs of $y_t(x)$ and $y_{\bar{h}}(x)$ in Figure 2 can only intersect for x equal to

$$-h_k^- \mathrm{E} \left(-k\pi, -\frac{2k-1}{2}\right)\pi \text{ where } k = 1, 2, ..., n-1 \text{ and } \left(\frac{2n-1}{2}\right)\pi < r < n\pi.$$

If $D^+(\bar{x}_0, \mathrm{E}) = 0$, then E is denoted by E_o and we have both

$$D^+(\bar{x}_0, \mathrm{E}_0) = 0 \text{ and } D_{\bar{x}}^{\pm}(\bar{x}_0, \mathrm{E}_0) = 0$$

Equation (5) implies that

$$\mathrm{E}_0 = n\pi - \csc \bar{x}_0$$

and equation (4) yields

$$n\pi - \bar{x}_0 - (n\pi - \mathrm{E}_0) \cos \bar{x}_0 = 0$$

Then, equation (3) implies that $|\bar{y}_t(\bar{x}_0)| = |\bar{y}_{\bar{h}}(\bar{x}_0)|$ and, after transforming back to the (x, y) coordinate system, we see that $y_t(x) = \tan x$ and $y_{\bar{h}}(x) = -\frac{\sqrt{r^2 - x^2}}{x}$ are tangent to each other at $-h_n^- = x_0 = n\pi + \bar{x}_0$.

Finally, if $D^+(\bar{x}_0, \mathrm{E}) < 0$, then, since $D^+(0, \mathrm{E}) = \mathrm{E} > 0$ and $D^+\left(\frac{\pi}{2}, \mathrm{E}\right) = n\pi - \frac{\pi}{2} > 0$, the intermediate value theorem implies that there exist $\bar{x}$ values $\bar{a}$ and $\bar{b}$ such that $0 < \bar{a} < \bar{x}_0 < \bar{b} < \frac{\pi}{2}$ and $D^+(\bar{a}, \mathrm{E}) = 0$, $D^+(\bar{b}, \mathrm{E}) = 0$. Therefore, besides the n-1 horizontal tangents of Case 1, there are two more, one at $x = -h_n^- = -a = -n\pi + \bar{a}$ and the other at $-h_{n+1}^{-1} = -b = -n\pi + \bar{b}$.

If $-\frac{\pi}{2} \le \bar{x} \le 0$, we consider the function $D^-(\bar{x}, \mathrm{E}) = n\pi - \bar{x} + (n\pi - \mathrm{E}) \cos \bar{x}$ and its partial derivative.

$D_{\bar{x}}^-(\bar{x}, \mathrm{E}) = -1 - (n\pi - \mathrm{E}) \sin \bar{x}$ finding $-\bar{x}_0 \mathrm{E} \left(-\frac{\pi}{2}, 0\right)$ such that $D_{\bar{x}}^-(-\bar{x}_0, \mathrm{E}) = 0$ and $D_{\bar{x}\bar{x}}^-(-\bar{x}_0, \mathrm{E}) < 0$.

Again, we consider three cases:

(1) $D^{-}(-\overline{x}_0, \mathrm{E}) > 0$ (2) $D^{-}(-\overline{x}_0, \mathrm{E}) = 0$ (3) $D^{-}\left(-\overline{x}_0, \mathrm{E}\right) > 0$

and by similar arguments as above, we find horizontal tangents at

$$-h_k^{-}\,\mathrm{E}\left(\left(\frac{2k-1}{2}\right)\pi, k\pi\right) \text{ where } k = 1, 2,..., n-1 \text{ and } \left(\frac{2n-1}{2}\right)\pi < r < n\pi \text{ in case 1.}$$

In case 2 we find, besides these n-1 values of x, one more at $h_n^{-} = x_0 = n\pi - \overline{x}_0$.

In case 3 we find two more at $h_n^{-} = a = n\pi - \overline{a}$ and $h_{n+1}^{-} = b = n\pi - \overline{b}$.

Finally, if we apply the second derivative test to functions defined by equation (1) or (2), we can prove that the graph traced out by $\overrightarrow{r}(x)$ has inflection points at $x = h_n^{-}$ and $x = -h_n^{-}$ if $r = n\pi - \mathrm{E}_0$. Also, if $n\pi - \mathrm{E} = r$ where $\mathrm{E} < \mathrm{E}_0$ then this graph has, in addition to all the other horizontal tangents mentioned above, local minima at $x = h_n^{-}$ and $x = -h_n^{-}$ as well as the local maxima at $x = h_n^{-}$ and $x = -h_n^{-}$. The existence of horizontal tangents that coincide if $x = \pm n\pi$ is easily verified. This proves the theorem.

The above theorem was proven with the hypothesis that $\frac{2n-1}{2}\pi < r \leq n\pi$.

A similar theorem can be proven if we assume that $(n-1)\pi < r \leq \frac{2(n-1)}{2}\pi$, but in this case, we have three cases for vertical tangents instead of horizontal ones. These cases depend on whether or not the graphs of $y_t(x) = \tan x$ and $y_v(x) = \frac{x}{\sqrt{r^2 - x^2}}$ do not intersect, are tangent to each other at one point, or intersect at two points in the intervals where $(n-1)\pi < |x| \leq \frac{2(n-1)}{2}\pi$ in some neighborhood of $\pm\frac{2n-1}{2}\pi$. The proof is analogous to the one given above.

We now give two examples illustrating the last two cases in part II of the theorem. Several examples of the first case have been given in the basic theory of hyperpolar circles (Glaser, 1992).

Example 1. The hyperpolar image of the circle $x^2 + y^2 = 9 = 3^2$ illustrates case 3. The position vector of this curve is

$$\overrightarrow{r}(x) = \exp\sqrt{9 - x^2}\,\left\langle \cos x, \sin x \right\rangle$$

and its derivative is

$$\vec{r}\,'(x) = \exp\left[\pm\sqrt{9-x^2}\right]\left\langle \mp\frac{x\cos x}{\sqrt{9-x^2}} - \sin x,\ \mp\frac{x\sin x}{\sqrt{9-x^2}} + \cos x \right\rangle$$

To find the x values for which there are vertical tangents, we solve the equations $\tan x = \mp\frac{x}{\sqrt{9-x^2}}$ when $y = \pm\sqrt{9-x^2}$ respectively, obtaining $x = v_0^{+} = 0^R$, $\pm v_1^{+} \cong 128° \cong 2.334^R$ and $v_0^{-} = 0^R$.

To find the x values for which there are horizontal tangents, we solve

$\tan x = \pm\frac{\sqrt{9-x^2}}{x}$ when $y = \pm\sqrt{9-x^2}$ respectively, obtaining
$x = \pm h_1^{+} \cong \pm 67° \cong \pm 1.169^R$ and $\pm h_2^{-} = \pm 168°\ 20' = \pm 2.937^R$.

Evaluating the position vector at these values of x and using the symbols V and H to denote points of vertical and horizontal tangents respectively, we find

1. For $y = \sqrt{9-x^2}$, the three points
   $v_0^{+} = (20.086, 0.000)$
   $v_{\pm 1}^{+} = (\pm 4.144, \pm 5,332)$
   $H_{\pm 1}^{+} = (6.190, \pm 14,585)$

2. For $y = -\sqrt{9-x^2}$, the three points
   $v_0^{+} = (0.0496, 0.000)$
   $H_{\pm 1}^{-} = (0.2207, \pm 0.1204)$
   $H_{\pm 2}^{-} = (0.5308, \pm 0.1063)$

The approximation method used to compute the values of x for which there are vertical and horizontal tangents to the graph traced out by $\vec{r}(x)$ is described in some mathematical handbooks (Burington, 1955) and in this example we have computed the values with accuracy up to two significant digits after the decimal place.

The graph of the curve is best displayed in two parts, one of which lies outside of the hyperpolar circle and is called the "macrograph" because coordinates of points lying on it are usually very large in absolute value. The other part lies inside the hyperpolar circle and is called the "micrograph" because its coordinates are relatively small.

Figures 3 and 4 show respectively both the macrograph and the micrograph of the hyperpolar image of $x^2 + y^2 = 9$. It can be shown that the points V_1^{+} and V_{-1}^{+} lie on the curve C_v: $\ln\left(u_2 + v_2\right) = -2\frac{u}{v}\tan^{-1}\left(\frac{v}{u}\right)$ and that the points H_1^{+}, H_{-1}^{+},

Figure 3. The macrograph of the hyperpolar image of $x^2 + y^2 = 9$.

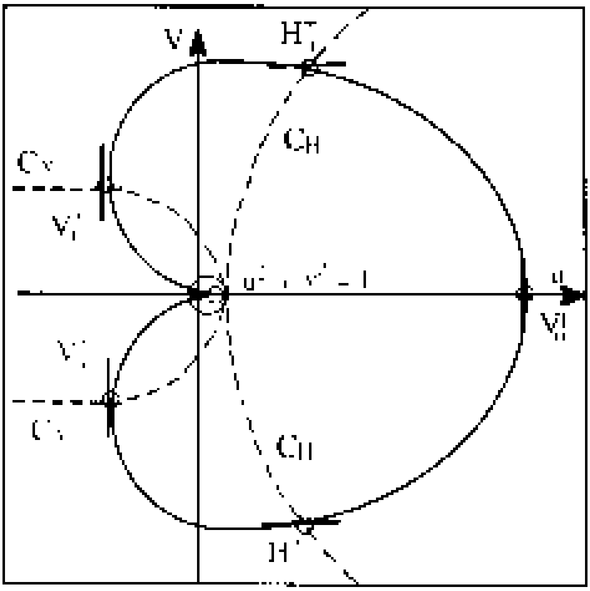
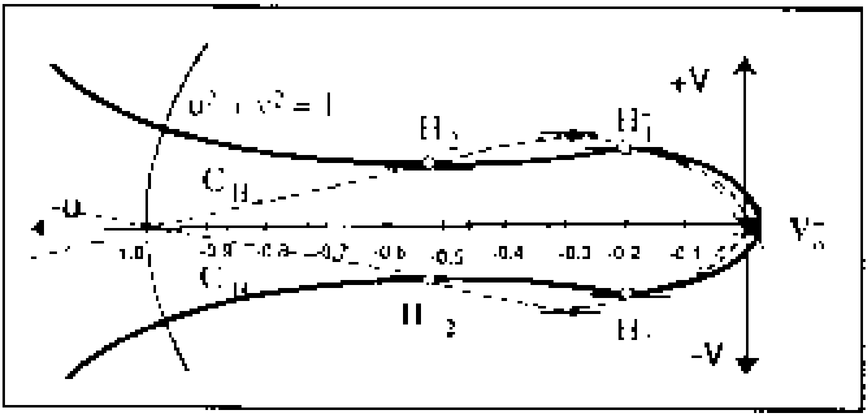


Figure 4. The micrograph of the hyperpolar image of $x^2 + y^2 = 9$



H_1^-, H_2^-, H_{-1}^-, and H_{-2}^- all lie on the curve C_H: $\ln(u^2 + v^2) = 2\frac{v}{u}\tan^{-1}\left(\frac{v}{u}\right)$. This is a general property of the hyperpolar images of Cartesian circles of the form $x^2 + y^2 = r^2$ and the curves C_v and C_H are also shown in Figures 3 and 4.

Example 2. The hyperpolar image of the circle $x^2 + y^2 = (2.97)^2$ illustrates case 2 of part II of the theorem. The micrograph has two inflection points whose coordinates are calculated below with an accuracy of two significant digits beyond the decimal place.

Applying an approximation method (Burrington, 1955) we solve $\tan x = \pm\frac{x}{\sqrt{(2.97)^2 - x^2}}$ when $y = \pm\sqrt{(2.97)^2 - x^2} = (-0.3418, \pm 0.1199)$ respectively, obtaining $x = V_0^+ = 0, \pm V_1^+ \cong \pm 129° = \pm 2.2515^R$ and $x = v_0^- = 0$ for the x values for which there are vertical tangents to the graph of

$$\vec{r}(x) = \exp\left[\pm\sqrt{(2.97)^2 - x^2}\right]\langle \cos x, \sin x\rangle$$

Solving the equations

$$\tan x = \pm\frac{x}{\sqrt{(2.97)^2 - x^2}} \text{ when } y = \pm\sqrt{(2.97)^2 - x^2}$$

respectively, we obtain $x = \pm h_1^+ \cong \pm 66° \cong 1.152^R$ and $x = \pm h_1^- \cong \pm 160°\ 20' \cong \pm 2.7985$ for the x values for which there are horizontal tangents to the graph of $\vec{r}(x)$.

Evaluating the position vector at these values of x and using the same symbolism as in Example 1 above, we find on the macrograph the points

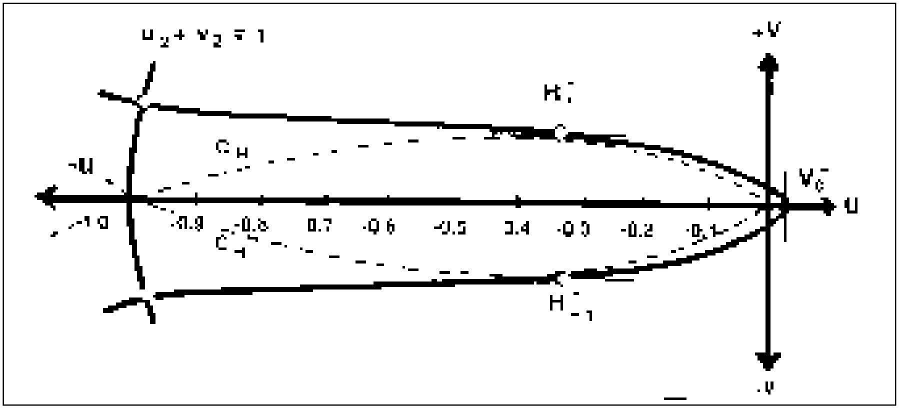
$$V_0^+ = (16.418,\ 0.000)$$
$$V_{\pm 1}^+ = (-4.467, \pm 5.324)$$
$$H_{\pm 1}^- = (-0.3418, \pm 13.969)\ .$$

On the micrograph we find

$$V_0^- = (0.0609,\ 0.0000)$$
$$H_{\pm 1}^- = (-0.3418, \pm 0.1199)$$

Since the macrograph of $\vec{r}(x)$ is just a variation of the one shown in Example 1 above, we only present the micrograph in Figure 5.

Figure 5. The micrograph of the hyperpolar image of $x^2 + y^2 = (2.97)^2$.



To show that $H^-_{\pm 1}$ are inflection points, we calculate by implicit differentiation in equation (1) the second derivative obtaining

$$\left[2u\tan^{-1}\left(\frac{v}{u}\right) + v\ln\left(u^2 + v^2\right)\right]\frac{d^2v}{du^2} + \left[\ln\left(u^2 + v^2\right) + 2\right]\left[\left(\frac{dv}{du}\right)^2 + 1\right] = 0$$

At $H^-_{\pm 1}$ we have $\frac{dv}{du} = 0$ and also $\ln\left(u^2 + v^2\right) + 2 = 0$.

$\Big($ Actually, we find $\ln\left[(-0.3418)^2 + (\pm 0.1199)^2\right] + 2 \cong 0\Big)$ but the factor $2u\tan^{-1}\left(\frac{v}{u}\right) + v\ln\left(u^2 + v^2\right)$ is positive at H^-_1 and negative at H^-_{-1}. Hence, $\left.\frac{d^2v}{du^2}\right|_{H^-_{\pm 1}} = 0$ and $H^-_{\pm 1}$ are inflection points.

Conclusion

The above theorem completes the theory of the properties of the images of Cartesian circles with the center at the origin. It must be recalled that if the center is not at the origin; we perform a translation of axes in the Cartesian plane that places the center at the origin of the new $(\bar{x}, \bar{y})$ coordinate system and then apply the hyperpolar transformation. The graph will then have the same properties as those stated in the theorem in the resulting $(\bar{u}, \bar{v})$ system. In order to change back to the standard (u,v) system, we must perform a rotation followed by a change of scale as has already been shown in the basic theory of hyperpolar circles (Glaser, 1992). These two transformations will change the location of vertical and horizontal tangents, but the shape of the graph remains the same.

References

Burington, R. S. , (1955). Handbook of mathematical tables and formulas. Sandusky, OH: Handbook Publisher Inc.

Glaser, F., (1992). Images of Cartesian lines and circles in the hyperpolar plane. The Cal Poly Scholar, Volume 5.