Homework 6 - 116
Solutions

X **45.** Use the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to show that

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

What can you conclude by comparing this result with that of Exercise 44?

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SOLUTION Using the Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$, we have for $|x| < 1$

$$\ln(1+x) - \ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-x)^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n + \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1+(-1)^{n-1}}{n} x^n$$

Since $1 + (-1)^{n-1} = 0$ for even n and $1 + (-1)^{n-1} = 2$ for odd n,

$$\ln(1+x) - \ln(1-x) = \sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1}.$$

Thus,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{2}{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.$$

Observe that this is the same series we found in Exercise 44; therefore,

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \tanh^{-1} x.$$

48. Let $F(x) = \int_0^x \frac{\sin t\, dt}{t}$. Show that

$$F(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

Evaluate $F(1)$ to three decimal places.

SOLUTION Divide the Maclaurin series for $\sin t$ by t to obtain

$$\frac{\sin t}{t} = \frac{1}{t} \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}.$$

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Integrating both sides of this equation and using term-by-term integration, we find

$$F(x) = \int_0^x \frac{\sin t}{t}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

For $x = 1$,

$$F(1) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!(2n+1)}.$$

This is an alternating series with $a_n = \frac{1}{(2n+1)!(2n+1)}$; therefore, the error incurred by using S_N to approximate the value of the definite integral is bounded by

$$\left| \int_0^1 \frac{\sin t}{t}\, dt - S_N \right| \le a_{N+1} = \frac{1}{(2N+3)!(2N+3)}.$$

To guarantee the error is at most 0.0005, we must choose N so that

$$\frac{1}{(2N+3)!(2N+3)} < 0.0005 \quad \text{or} \quad (2N+3)!(2N+3) > 2000.$$

For $N = 1$, $(2N+3)!(2N+3) = 5! \cdot 5 = 600 < 2000$ and for $N = 2$, $(2N+3)!(2N+3) = 7! \cdot 7 = 35280 > 2000$; thus, the smallest acceptable value for N is $N = 2$. The corresponding approximation is

$$S_2 = \sum_{n=0}^{2} \frac{(-1)^n}{(2n+1)!(2n+1)} = 1 - \frac{1}{3 \cdot 3!} + \frac{1}{5 \cdot 5!} = 0.946111111.$$

X 59. Which function has Maclaurin series $\sum_{n=0}^{\infty} (-1)^n 2^n x^n$?

SOLUTION We recognize that

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$$\sum_{n=0}^{\infty} (-1)^n 2^n x^n = \sum_{n=0}^{\infty} (-2x)^n$$

is the Maclaurin series for $\frac{1}{1-x}$ with x replaced by $-2x$. Therefore,

$$\sum_{n=0}^{\infty} (-1)^n 2^n x^n = \frac{1}{1-(-2x)} = \frac{1}{1+2x}.$$

X 60. $f(x) = \sin(x^2)\cos(x^2)$

SOLUTION Substituting x^2 for x in the Maclaurin series for $\sin x$ and $\cos x$, we find

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$$\begin{aligned}
\sin(x^2)\cos(x^2) &= \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \cdots\right)\left(1 - \frac{x^4}{2} + \frac{x^8}{24} - \frac{x^{12}}{720} + \cdots\right) \\
&= x^2 - \frac{x^6}{6} - \frac{x^6}{2} + \frac{x^{10}}{24} + \frac{x^{10}}{12} + \frac{x^{10}}{120} - \frac{x^{14}}{720} - \frac{x^{14}}{144} - \frac{x^{14}}{240} - \frac{x^{14}}{5040} + \cdots \\
&= x^2 - \frac{2}{3}x^6 + \frac{2}{15}x^{10} - \frac{4}{315}x^{14} + \cdots.
\end{aligned}$$