Homework 5 - 116
Solutions

38. Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$. *Hint:* Show that

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$$(1-x)^{-2} = \sum_{n=1}^{\infty} n x^{n-1}$$

SOLUTION Differentiate both sides of Eq. (1) to obtain

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Setting $x = \frac{1}{2}$ then yields

$$\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4.$$

Divide this equation by 2 to obtain

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

39. Give an example of a power series that converges for x in $[2, 6)$.

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SOLUTION The power series must be centered at $c = \frac{6+2}{2} = 4$, with radius of convergence $R = 2$. Consider the following series:

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{n 2^n}.$$

With $a_n = \frac{1}{n2^n}$,

$$r = \lim_{n\to\infty} \frac{n2^n}{(n+1)2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{2}.$$

The radius of convergence is therefore $R = r^{-1} = 2$, and the series converges absolutely for $|x-4| < 2$, or $2 < x < 6$. For the endpoint $x = 6$, the series becomes $\sum_{n=1}^{\infty} \frac{(6-4)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent harmonic series. For the endpoint $x = 2$, the series becomes $\sum_{n=1}^{\infty} \frac{(2-4)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the Leibniz Test. Therefore, the series converges for $2 \le x < 6$, as desired.

40. Prove that for $-1 < x < 1$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$

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$$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

SOLUTION To obtain the first expansion, substitute $-x$ for x in Eq. (1):

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

This expansion is valid for $|-x| < 1$, or $-1 < x < 1$.

Upon integrating both sides of the above equation, we find

$$\ln(1+x) = \int \frac{dx}{1+x} = \int \left(\sum_{n=0}^{\infty}(-1)^n x^n\right) dx.$$

Integrating the series term-by-term then yields

$$\ln(1+x) = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

To determine the constant C, set $x = 0$. Then $0 = \ln(1+0) = C$. Finally,

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}.$$

41. Use Exercise 40 to prove that

$$\ln\frac{3}{2} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$$

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Use your knowledge of alternating series to find an N such that the partial sum S_N approximates $\ln\frac{3}{2}$ to within an error of at most 10^{-3}. Confirm this using a calculator to compute both S_N and $\ln\frac{3}{2}$.

SOLUTION In the previous exercise we found that

$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}.$$

Setting $x = \frac{1}{2}$ yields:

$$\ln\frac{3}{2} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\left(\frac{1}{2}\right)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n} = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots$$

Note that the series for $\ln\frac{3}{2}$ is an alternating series with $a_n = \frac{1}{n2^n}$. The error in approximating $\ln\frac{3}{2}$ by the partial sum S_N is therefore bounded by

$$\left|\ln\frac{3}{2} - S_N\right| < a_{N+1} = \frac{1}{(N+1)2^{N+1}}.$$

To obtain an error of at most 10^{-3}, we must find an N such that

$$\frac{1}{(N+1)2^{N+1}} < 10^{-3} \quad \text{or} \quad (N+1)2^{N+1} > 1000.$$

For $N = 6$, $(N+1)2^{N+1} = 7\cdot 2^7 = 896 < 1000$, but for $N = 7$, $(N+1)2^{N+1} = 8\cdot 2^8 = 2048 > 1000$; hence, the smallest value for N is $N = 7$. The corresponding approximation is

$$S_7 = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \frac{1}{5\cdot 2^5} - \frac{1}{6\cdot 2^6} + \frac{1}{7\cdot 2^7} = 0.405803571.$$

Now, $\ln\frac{3}{2} = 0.405465108$, so

$$\left|\ln\frac{3}{2} - S_7\right| = 3.385 \times 10^{-4} < 10^{-3}.$$

47. Find a power series $P(x)$ satisfying the differential equation:

$$y'' - xy' + y = 0$$

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with initial condition $y(0) = 1$, $y'(0) = 0$. What is the radius of convergence of the power series?

SOLUTION Let $P(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$P'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad P''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Note that $P(0) = a_0$ and $P'(0) = a_1$; in order to satisfy the initial conditions $P(0) = 1$, $P'(0) = 0$, we must have $a_0 = 1$ and $a_1 = 0$. Now,

$$\begin{aligned} P''(x) - xP'(x) + P(x) &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} - n a_n + a_n\right] x^n. \end{aligned}$$

In order for this series to be equal to zero, the coefficient of x^n must be equal to zero for each n; thus, $2a_2 + a_0 = 0$ and $(n+2)(n+1)a_{n+2} - (n-1)a_n = 0$, or

$$a_2 = -\frac{1}{2}a_0 \quad \text{and} \quad a_{n+2} = \frac{n-1}{(n+2)(n+1)}a_n.$$

Starting from $a_1 = 0$, we calculate

$$a_3 = \frac{1-1}{(3)(2)}a_1 = 0;$$

$$a_5 = \frac{2}{(5)(4)}a_3 = 0;$$

$$a_7 = \frac{4}{(7)(6)}a_5 = 0;$$

and, in general, all of the odd coefficients are zero. As for the even coefficients, we have $a_0 = 1$, $a_2 = -\frac{1}{2}$,

$$a_4 = \frac{1}{(4)(3)}a_2 = -\frac{1}{4!};$$

$$a_6 = \frac{3}{(6)(5)}a_4 = -\frac{3}{6!};$$

$$a_8 = \frac{5}{(8)(7)}a_6 = -\frac{15}{8!}$$

and so on. Thus,

$$P(x) = 1 - \frac{1}{2}x^2 - \frac{1}{4!}x^4 - \frac{3}{6!}x^6 - \frac{15}{8!}x^8 - \cdots$$

To determine the radius of convergence, treat this as a series in the variable x^2, and observe that

$$r = \lim_{k\to\infty} \left|\frac{a_{2k+2}}{a_{2k}}\right| = \lim_{k\to\infty} \frac{2k-1}{(2k+2)(2k+1)} = 0.$$

Thus, the radius of convergence is $R = r^{-1} = \infty$.