

# Home work 4 Solutions

3

30. Prove that if  $\sum a_n$  converges absolutely, then  $\sum a_n^2$  also converges. Then show by giving a counterexample that  $\sum a_n^2$  need not converge if  $\sum a_n$  is only conditionally convergent.

SOLUTION Suppose the series  $\sum a_n$  converges absolutely. Because  $\sum |a_n|$  converges, we know that

$$\lim_{n \rightarrow \infty} |a_n| = 0.$$

Therefore, there exists a positive integer  $N$  such that  $|a_n| < 1$  for all  $n \geq N$ . It then follows that for  $n \geq N$ ,

$$0 \leq a_n^2 = |a_n|^2 = |a_n| \cdot |a_n| < |a_n| \cdot 1 = |a_n|.$$

By the Comparison Test we can then conclude that  $\sum a_n^2$  also converges.

Consider the series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ . This series converges by the Leibniz Test, but the corresponding positive series is a divergent  $p$ -series; that is,  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$  is conditionally convergent. Now,  $\sum_{n=1}^{\infty} a_n^2$  is the divergent harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n}$ . Thus,  $\sum a_n^2$  need not converge if  $\sum a_n$  is only conditionally convergent.

26.  $\sum_{n=1}^{\infty} na_n$

SOLUTION Let  $b_n = na_n$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} \left| \frac{a_{n+1}}{a_n} \right| = 1 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series  $\sum_{n=1}^{\infty} na_n$  converges by the Ratio Test.

$$27. \sum_{n=1}^{\infty} n^3 a_n$$

SOLUTION Let  $b_n = n^3 a_n$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^3 \left| \frac{a_{n+1}}{a_n} \right| = 1^3 \cdot \frac{1}{3} = \frac{1}{3} < 1.$$

Therefore, the series  $\sum_{n=1}^{\infty} n^3 a_n$  converges by the Ratio Test.

$$28. \sum_{n=1}^{\infty} 2^n a_n$$

SOLUTION Let  $b_n = 2^n a_n$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1}}{2^n} \left| \frac{a_{n+1}}{a_n} \right| = 2 \cdot \frac{1}{3} = \frac{2}{3} < 1.$$

Therefore, the series  $\sum_{n=1}^{\infty} 2^n a_n$  converges by the Ratio Test.

$$29. \sum_{n=1}^{\infty} 3^n a_n$$

SOLUTION Let  $b_n = 3^n a_n$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{3^{n+1}}{3^n} \left| \frac{a_{n+1}}{a_n} \right| = 3 \cdot \frac{1}{3} = 1.$$

Therefore, the Ratio Test is inconclusive for the series  $\sum_{n=1}^{\infty} 3^n a_n$ .

$$30. \sum_{n=1}^{\infty} 4^n a_n$$

SOLUTION Let  $b_n = 4^n a_n$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \frac{4^{n+1}}{4^n} \left| \frac{a_{n+1}}{a_n} \right| = 4 \cdot \frac{1}{3} = \frac{4}{3} > 1.$$

Therefore, the series  $\sum_{n=1}^{\infty} 4^n a_n$  diverges by the Ratio Test.

$$31. \sum_{n=1}^{\infty} a_n^2$$

SOLUTION Let  $b_n = a_n^2$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|^2 = \left( \frac{1}{3} \right)^2 = \frac{1}{9} < 1.$$

Therefore, the series  $\sum_{n=1}^{\infty} a_n^2$  converges by the Ratio Test.

32. Assume that  $\left| \frac{a_{n+1}}{a_n} \right|$  converges to  $\rho = 4$ . Does  $\sum_{n=1}^{\infty} a_n^{-1}$  converge (assume that  $a_n \neq 0$  for all  $n$ )?

SOLUTION Let  $b_n = a_n^{-1}$ . Then

$$\rho = \lim_{n \rightarrow \infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|} = \frac{1}{4} < 1.$$

Therefore, the series  $\sum_{n=1}^{\infty} a_n^{-1}$  converges by the Ratio Test.

(2)