46. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. *Hint:* Find constants A, B, and C such that

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{(n+1)} + \frac{C}{n+2}$$

① Homework 3

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SOLUTION By partial fraction decomposition

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2};$$

clearing denominators then gives

$$1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1).$$

Setting $n = 0$ now yields $A = \frac{1}{2}$, while setting $n = -1$ yields $B = -1$ and setting $n = -2$ yields $C = \frac{1}{2}$. Thus,

$$\frac{1}{n(n+1)(n+2)} = \frac{\frac{1}{2}}{n} - \frac{1}{n+1} + \frac{\frac{1}{2}}{n+2} = \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right).$$

The general term of the sequence of partial sums for the series on the right-hand side is

$$\begin{aligned} S_N &= \frac{1}{2}\left(1 - \frac{2}{2} + \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{4} + \frac{2}{5} + \frac{1}{6}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) \\ &\quad + \cdots + \frac{1}{2}\left(\frac{1}{N-2} - \frac{2}{N-1} + \frac{1}{N}\right) + \frac{1}{2}\left(\frac{1}{N-1} - \frac{2}{N} + \frac{1}{N+1}\right) + \frac{1}{2}\left(\frac{1}{N} - \frac{2}{N+1} + \frac{1}{N+2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N+1} + \frac{1}{N+2}\right). \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{N+1} + \frac{1}{N+2}\right) = \frac{1}{4}.$$

47. Show that if a is a positive integer, then

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \frac{1}{a}\left(1 + \frac{1}{2} + \cdots + \frac{1}{a}\right)$$

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SOLUTION By partial fraction decomposition

$$\frac{1}{n(n+a)} = \frac{A}{n} + \frac{B}{n+a};$$

clearing the denominators gives

$$1 = A(n+a) + Bn.$$

Setting $n = 0$ then yields $A = \frac{1}{a}$, while setting $n = -a$ yields $B = -\frac{1}{a}$. Thus,

$$\frac{1}{n(n+a)} = \frac{\frac{1}{a}}{n} - \frac{\frac{1}{a}}{n+a} = \frac{1}{a}\left(\frac{1}{n} - \frac{1}{n+a}\right),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \sum_{n=1}^{\infty} \frac{1}{a}\left(\frac{1}{n} - \frac{1}{n+a}\right).$$

For $N > a$, the Nth partial sum is

$$S_N = \frac{1}{a}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{a}\right) - \frac{1}{a}\left(\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3} + \cdots + \frac{1}{N+a}\right).$$

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+a)} = \lim_{N\to\infty} S_N = \frac{1}{a}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{a}\right).$$

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45. Show that $\sum_{n=2}^{\infty}(\ln n)^{-2}$ diverges. *Hint:* Show that for x sufficiently large, $\ln x < x^{1/2}$.

SOLUTION Using L'Hôpital's Rule,

$$\lim_{x\to\infty}\frac{x^{1/2}}{\ln x}=\lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{1}{2}x^{1/2}=\infty;$$

thus, there exists an integer N such that

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$$\frac{n^{1/2}}{\ln n}>1 \quad\text{or}\quad n^{1/2}>\ln n$$

for all $n\geq N$. Hence,

$$\frac{1}{\ln n}>\frac{1}{n^{1/2}} \quad\text{and}\quad \frac{1}{(\ln n)^2}>\frac{1}{n}$$

for all $n\geq N$. The harmonic series diverges, so the series $\sum_{n=N}^{\infty}\frac{1}{n}$ also diverges. By the Comparison Test we can therefore conclude that the series $\sum_{n=N}^{\infty}\frac{1}{(\ln n)^2}$ diverges. It follows that the series $\sum_{n=2}^{\infty}(\ln n)^{-2}$ also diverges.

46. For which a does $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^a}$ converge?

SOLUTION First consider the case $a>0$ but $a\neq 1$. Let $f(x)=\frac{1}{x(\ln x)^a}$. This function is continuous, positive and decreasing for $x\geq 2$, so the Integral Test applies. Now,

$$\int_2^{\infty}\frac{dx}{x(\ln x)^a}=\lim_{R\to\infty}\int_2^R\frac{dx}{x(\ln x)^a}=\lim_{R\to\infty}\int_{\ln 2}^{\ln R}\frac{du}{u^a}=\frac{1}{1-a}\lim_{R\to\infty}\left(\frac{1}{(\ln R)^{a-1}}-\frac{1}{(\ln 2)^{a-1}}\right).$$

Because

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$$\lim_{R\to\infty}\frac{1}{(\ln R)^{a-1}}=\begin{cases}\infty, & 0<a<1\\ 0, & a>1\end{cases}$$

we conclude the integral diverges when $0<a<1$ and converges when $a>1$. Therefore

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^a}\text{ converges for } a>1 \text{ and diverges for } 0<a<1.$$

Next, consider the case $a=1$. The series becomes $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$. Let $f(x)=\frac{1}{x\ln x}$. For $x\geq 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u=\ln x$, $du=\frac{1}{x}\,dx$, we find

$$\int_2^{\infty}\frac{dx}{x\ln x}=\lim_{R\to\infty}\int_2^R\frac{dx}{x\ln x}=\lim_{R\to\infty}\int_{\ln 2}^{\ln R}\frac{du}{u}=\lim_{R\to\infty}(\ln(\ln R)-\ln(\ln 2))=\infty.$$

The integral diverges; hence, the series also diverges.

Finally, consider the case $a<0$. Let $b=-a>0$ so the series becomes $\sum_{n=2}^{\infty}\frac{(\ln n)^b}{n}$. Since $\ln n>1$ for all $n\geq 3$, it follows that

$$(\ln n)^b>1 \quad\text{so}\quad \frac{(\ln n)^b}{n}>\frac{1}{n}.$$

The series $\sum_{n=3}^{\infty}\frac{1}{n}$ diverges, so by the Comparison Test we can conclude that $\sum_{n=3}^{\infty}\frac{(\ln n)^b}{n}$ also diverges. Consequently, $\sum_{n=2}^{\infty}\frac{(\ln n)^b}{n}$ diverges. Thus,

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^a}\text{ diverges for } a<0.$$

To summarize:

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^a}\text{ converges if } a>1 \text{ and diverges if } a\leq 1.$$

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47. For which a does $\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$ converge?

SOLUTION First consider the case $a > 1$. For $n \geq 3$, $\ln n > 1$ and

$$\frac{1}{n^a \ln n} < \frac{1}{n^a}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^a}$ is a p-series with $p = a > 1$, so it converges; hence, $\sum_{n=3}^{\infty} \frac{1}{n^a}$ also converges. By the Comparison Test we can therefore conclude that the series $\sum_{n=3}^{\infty} \frac{1}{n^a \ln n}$ converges, which implies the series $\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$ also converges.

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For $a \leq 1$, $n^a \leq n$ so

$$\frac{1}{n^a \ln n} \geq \frac{1}{n \ln n}$$

for $n \geq 2$. Let $f(x) = \frac{1}{x \ln x}$. For $x \geq 2$, this function is continuous, positive and decreasing, so the Integral Test applies. Using the substitution $u = \ln x$, $du = \frac{1}{x}\,dx$, we find

$$\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_2^R \frac{dx}{x \ln x} = \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u} = \lim_{R\to\infty} (\ln(\ln R) - \ln(\ln 2)) = \infty.$$

The integral diverges; hence, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ also diverges. By the Comparison Test we can therefore conclude that the series $\sum_{n=2}^{\infty} \frac{1}{n^a \ln n}$ diverges.

To summarize,

$$\sum_{n=2}^{\infty} \frac{1}{n^a \ln n} \text{ converges for } a > 1 \text{ and diverges for } a \leq 1.$$