Hmk 2 - solutions

75. Prove that the following sequence is bounded and increasing. Then find its limit:

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$$a_1 = \sqrt{5}, \quad a_2 = \sqrt{5+\sqrt{5}}, \quad a_3 = \sqrt{5+\sqrt{5+\sqrt{5}}}, \ldots$$

SOLUTION Notice that this sequence is defined recursively by the formula:

$$a_{n+1} = \sqrt{5+a_n}.$$

First, let's show that the sequence is bounded. All the terms in the sequence are positive, so 0 is a lower bound. Now, $a_1 = \sqrt{5} < 3$. If we suppose that $a_n < 3$ for some n, it then follows that

$$a_{n+1} = \sqrt{5+a_n} < \sqrt{5+3} = \sqrt{8} < \sqrt{9} = 3.$$

Thus, by mathematical induction, $a_n < 3$ for all n, and 3 is an upper bound. Next, let's show that the sequence is increasing. Observe that $a_1 = \sqrt{5} \approx 2.236$ and $a_2 = \sqrt{5+\sqrt{5}} \approx 2.690$. Thus, $a_2 > a_1$. If we suppose that $a_n > a_{n-1}$ for some n, it then follows that

$$a_{n+1} = \sqrt{5+a_n} > \sqrt{5+a_{n-1}} = a_n;$$

hence, by mathematical induction, $a_{n+1} > a_n$ for all n.

Now, since $\{a_n\}$ is increasing with upper bound, this sequence is convergent. Let $\lim_{n\to\infty} a_n = L$. Then, by Exercise 68, $\lim_{n\to\infty} a_{n+1} = L$ as well. It follows that

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5+a_n} = \sqrt{\lim_{n\to\infty}(5+a_n)} = \sqrt{5+\lim_{n\to\infty} a_n} = \sqrt{5+L}.$$

That is,

$$L^2 = 5 + L$$

$$L^2 - L - 5 = 0 \Rightarrow L_{1,2} = \frac{1 \pm \sqrt{1+20}}{2}.$$

Since $a_n \geq 0$ for all n, the appropriate solution is:

$$\lim_{n\to\infty} a_n = \frac{1+\sqrt{21}}{2}.$$

77. Find the limit of the sequence

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$$c_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$$

Hint: Show that

$$\frac{n}{\sqrt{n^2+n}} \leq c_n \leq \frac{n}{\sqrt{n^2+1}}$$

SOLUTION Since each of the n terms in the sum defining c_n is not smaller than $\frac{1}{\sqrt{n^2+n}}$ and not larger than $\frac{1}{\sqrt{n^2+1}}$ we obtain the following inequalities:

$$c_n \geq \underbrace{\frac{1}{\sqrt{n^2+n}} + \cdots + \frac{1}{\sqrt{n^2+n}}}_{n \text{ terms}} = n \cdot \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}};$$

$$c_n \leq \underbrace{\frac{1}{\sqrt{n^2+1}} + \cdots + \frac{1}{\sqrt{n^2+1}}}_{n \text{ terms}} = n \cdot \frac{1}{\sqrt{n^2+1}} = \frac{n}{\sqrt{n^2+1}}.$$

Thus,

$$\frac{n}{\sqrt{n^2+n}} \leq c_n \leq \frac{n}{\sqrt{n^2+1}}.$$

We now compute the limits of the two sequences:

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{\sqrt{n^2+1}}{n}}=\lim_{n\to\infty}\frac{1}{\frac{\sqrt{n^2+1}}{\sqrt{n^2}}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1;$$

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{\frac{n}{n}}{\frac{\sqrt{n^2+n}}{n}}=\lim_{n\to\infty}\frac{1}{\frac{\sqrt{n^2+n}}{\sqrt{n^2}}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1.$$

By the Squeeze Theorem we conclude that:

$$\lim_{n\to\infty}c_n=1.$$

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80. Given positive numbers $a_1 < b_1$, define two sequences recursively by

$$a_{n+1}=\sqrt{a_nb_n},\qquad b_n=\frac{a_n+b_n}{2}$$

(a) Show that $a_n \le b_n$ for all n (Figure 1).

(b) Show that $\{a_n\}$ is increasing and $\{b_n\}$ is decreasing.

(c) Show that

$$b_{n+1}-a_{n+1}\le\frac{b_n-a_n}{2}$$

Prove that both $\{a_n\}$ and $\{b_n\}$ converge and have the same limit. This limit, denoted AGM(a_1, b_1), is called the **arithmetic-geometric mean** of a_1 and b_1. See Figure 1.

(d) Estimate AGM$(1, \sqrt{2})$ to three decimal places.

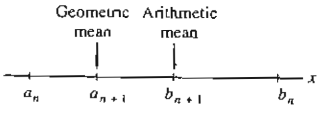


FIGURE 1

(a) Examine the following:

$$b_{n+1}-a_{n+1}=\frac{a_n+b_n}{2}-\sqrt{a_nb_n}=\frac{a_n+b_n-2\sqrt{a_nb_n}}{2}=\frac{\left(\sqrt{a_n}\right)^2-2\sqrt{a_n}\sqrt{b_n}+\left(\sqrt{b_n}\right)^2}{2}$$

$$=\frac{\left(\sqrt{a_n}-\sqrt{b_n}\right)^2}{2}\ge 0.$$

We conclude that $b_{n+1}\ge a_{n+1}$ for all $n > 1$. By the given information $b_1 > a_1$; hence, $b_n \ge a_n$ for all n.

(b) By part (a), $b_n \ge a_n$ for all n, so

$$a_{n+1}=\sqrt{a_nb_n}\ge\sqrt{a_n\cdot a_n}=\sqrt{a_n^2}=a_n$$

for all n. Hence, the sequence $\{a_n\}$ is increasing. Moreover, since $a_n \le b_n$ for all n,

$$b_{n+1}=\frac{a_n+b_n}{2}\le\frac{b_n+b_n}{2}=\frac{2b_n}{2}=b_n$$

for all n; that is, the sequence $\{b_n\}$ is decreasing.

(c) Since $\{a_n\}$ is increasing, $a_{n+1}\ge a_n$. Thus,

$$b_{n+1}-a_{n+1}\le b_{n+1}-a_n=\frac{a_n+b_n}{2}-a_n=\frac{a_n+b_n-2a_n}{2}=\frac{b_n-a_n}{2}.$$

Now, by part (a), $a_n \le b_n$ for all n. By part (b), $\{b_n\}$ is decreasing. Hence $b_n \le b_1$ for all n. Combining the two inequalities we conclude that $a_n \le b_1$ for all n. That is, the sequence $\{a_n\}$ is increasing and bounded $(0 \le a_n \le b_1)$. By the Theorem on Bounded Monotonic Sequences we conclude that $\{a_n\}$ converges. Similarly, since $\{a_n\}$ is increasing, $a_n \ge a_1$ for all n. We combine this inequality with $b_n \ge a_n$ to conclude that $b_n \ge a_1$ for all n. Thus, $\{b_n\}$ is decreasing and bounded $(a_1 \le b_n \le b_1)$; hence this sequence converges.

To show that $\{a_n\}$ and $\{b_n\}$ converge to the same limit, note that

$$b_n-a_n\le\frac{b_{n-1}-a_{n-1}}{2}\le\frac{b_{n-2}-a_{n-2}}{2^2}\le\cdots\le\frac{b_1-a_1}{2^{n-1}}.$$

Thus,

$$\lim_{n\to\infty}(b_n-a_n)=(b_1-a_1)\lim_{n\to\infty}\frac{1}{2^{n-1}}=0.$$

(d) We have

$$a_{n+1} = \sqrt{a_n b_n}, \quad a_1 = 1; \quad b_{n+1} = \frac{a_n + b_n}{2}, \quad b_1 = \sqrt{2}$$

Computing the values of a_n and b_n until the first three decimal digits are equal in successive terms, we obtain:

$$a_2 = \sqrt{a_1 b_1} = \sqrt{1 \cdot \sqrt{2}} = 1.1892$$

$$b_2 = \frac{a_1 + b_1}{2} = \frac{1 + \sqrt{2}}{2} = 1.2071$$

$$a_3 = \sqrt{a_2 b_2} = \sqrt{1.1892 \cdot 1.2071} = 1.1981$$

$$b_3 = \frac{a_2 + b_2}{2} = \frac{1.1892 \cdot 1.2071}{2} = 1.1981$$

$$a_4 = \sqrt{a_3 b_3} = 1.1981$$

$$b_4 = \frac{a_3 + b_3}{2} = 1.1981$$

Thus,

$$AGM\left(1, \sqrt{2}\right) \approx 1.198.$$

3

82. The nth harmonic number is the number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Let $a_n = H_n - \ln n$.

(a) Show that $a_n \geq 0$ for $n \geq 1$. *Hint:* Show that $H_n \geq \int_1^{n+1} \frac{dx}{x}$.

(b) Show that $\{a_n\}$ is decreasing by interpreting $a_n - a_{n+1}$ as an area.

(c) Prove that $\lim_{n\to\infty} a_n$ exists. This limit, denoted γ and known as *Euler's Constant*, appears in many areas of mathematics, including analysis and number theory. It has been calculated to more than 100 million decimal places, but it is still not known if γ is an irrational number. The first 10 digits are $\gamma \approx 0.5772156649$.

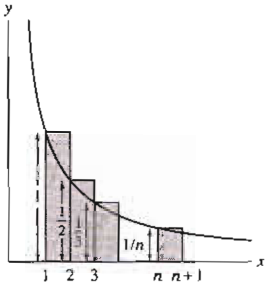
SOLUTION

(a) Since the function $y = \frac{1}{x}$ is decreasing, the left endpoint approximation to the integral $\int_1^{n+1} \frac{dx}{x}$ is greater than this integral; that is,

$$1 \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \cdots + \frac{1}{n} \cdot 1 \geq \int_1^{n+1} \frac{dx}{x}$$

or

$$H_n \geq \int_1^{n+1} \frac{dx}{x}.$$



Moreover, since the function $y = \frac{1}{x}$ is positive for $x > 0$, we have:

$$\int_1^{n+1} \frac{dx}{x} \geq \int_1^{n} \frac{dx}{x}.$$

Thus,

$$H_n \geq \int_1^{n} \frac{dx}{x} = \ln x\Big|_1^n = \ln n - \ln 1 = \ln n,$$

and

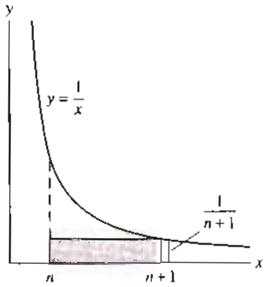
$$a_n = H_n - \ln n \geq 0 \qquad \text{for all } n \geq 1.$$

(b) To show that $\{a_n\}$ is decreasing, we consider the difference $a_n - a_{n+1}$:

$$\begin{aligned} a_n - a_{n+1} &= H_n - \ln n - \big(H_{n+1} - \ln(n+1)\big) = H_n - H_{n+1} + \ln(n+1) - \ln n \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right) + \ln(n+1) - \ln n \\ &= -\frac{1}{n+1} + \ln(n+1) - \ln n. \end{aligned}$$

Now, $\ln(n+1) - \ln n = \int_n^{n+1} \frac{dx}{x}$, whereas $\frac{1}{n+1}$ is the right endpoint approximation to the integral $\int_n^{n+1} \frac{dx}{x}$. Recalling $y = \frac{1}{x}$ is decreasing, it follows that

$$\int_n^{n+1} \frac{dx}{x} \geq \frac{1}{n+1}$$



so

$$a_n - a_{n+1} \geq 0.$$

(c) By parts (a) and (b), $\{a_n\}$ is decreasing and 0 is a lower bound for this sequence. Hence $0 \leq a_n \leq a_1$ for all n. A monotonic and bounded sequence is convergent, so $\lim_{n\to\infty} a_n$ exists.