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Hmk 1 – 116

Solutions

66. Show that $\lim_{x\to 0} \frac{f(x)}{x^k} = 0$ for all k. *Hint:* Let $t = x^{-1}$ and apply the result of Exercise 65.

SOLUTION $\lim_{x\to 0} \frac{f(x)}{x^k} = \lim_{x\to 0} \frac{1}{x^k e^{1/x^2}}$. Let $t = 1/x$. As $x \to 0$, $t \to \infty$. Thus,

$$\lim_{x\to 0} \frac{1}{x^k e^{1/x^2}} = \lim_{t\to\infty} \frac{t^k}{e^{t^2}} = 0$$

by Exercise 65.

67. Show that $f'(0)$ exists and is equal to zero. Also, verify that $f''(0)$ exists and is equal to zero.

SOLUTION Working from the definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = 0$$

by the previous exercise. Thus, $f'(0)$ exists and is equal to 0. Moreover,

$$f'(x) = \begin{cases} e^{-1/x^2}\left(\frac{2}{x^3}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Now,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} e^{-1/x^2}\left(\frac{2}{x^4}\right) = 2\lim_{x\to 0} \frac{f(x)}{x^4} = 0$$

by the previous exercise. Thus, $f''(0)$ exists and is equal to 0.

68. Show that for $k \geq 1$ and $x \neq 0$,

$$f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$$

for some polynomial $P(x)$ and some exponent $r \geq 1$. Use the result of Exercise 66 to show that $f^{(k)}(0)$ exists and is equal to zero for all $k \geq 1$.

SOLUTION For $x \neq 0$, $f'(x) = e^{-1/x^2}\left(\frac{2}{x^3}\right)$. Here $P(x) = 2$ and $r = 3$. Assume $f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$. Then

$$f^{(k+1)}(x) = e^{-1/x^2}\left(\frac{x^3P'(x) + (2 - rx^2)P(x)}{x^{r+3}}\right)$$

which is of the form desired.

Moreover, from Exercise 67, $f'(0) = 0$. Suppose $f^{(k)}(0) = 0$. Then

$$f^{(k+1)}(0) = \lim_{x\to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x - 0} = \lim_{x\to 0} \frac{P(x)e^{-1/x^2}}{x^{r+1}} = P(0)\lim_{x\to 0} \frac{f(x)}{x^{r+1}} = 0.$$

99. Let R be the region under the graph of $y = (x+1)^{-1}$ for $0 \le x < \infty$. Which of the following quantities is finite?

(a) The area of R

(b) The volume of the solid obtained by rotating R about the x-axis

(c) The volume of the solid obtained by rotating R about the y-axis

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SOLUTION

(a) The area of R is

$$\int_0^\infty \frac{dx}{x+1} = \lim_{R\to\infty} \int_0^R \frac{dx}{x+1} = \lim_{R\to\infty} \ln|x+1| \Big|_0^R = \lim_{R\to\infty} \left(\ln(R+1) - \ln 1\right) = \infty.$$

Hence, the area of R is not finite.

(b) Using the Disk Method, the volume of the solid obtained by rotating R about the x-axis is

$$\pi \int_0^\infty \frac{dx}{(x+1)^2} = \pi \lim_{R\to\infty} \int_0^R \frac{dx}{(x+1)^2} = \pi \lim_{R\to\infty} -\frac{1}{x+1}\Big|_0^R = \pi \lim_{R\to\infty} \left(-\frac{1}{R+1} + 1\right) = \pi.$$

Hence, the volume of the solid obtained by rotating R about the x-axis is finite.

(c) Using the Shell Method, the volume of the solid obtained by rotating R about the y-axis is

$$2\pi \int_0^\infty \frac{x}{x+1}\, dx = 2\pi \lim_{R\to\infty} \int_0^R \frac{x\,dx}{x+1}.$$

Now,

$$\int_0^R \frac{x\,dx}{x+1} = \int_0^R \frac{(x+1)-1}{x+1}\, dx = \int_0^R \left(1 - \frac{1}{x+1}\right) dx = (x - \ln(x+1))\Big|_0^R$$
$$= R - (\ln(R+1) - \ln 1) = R - \ln(R+1).$$

Thus,

$$2\pi \lim_{R\to\infty} \int_0^R \frac{x\,dx}{x+1} = 2\pi \lim_{R\to\infty} (R - \ln(R+1)) = 2\pi \lim_{R\to\infty} R\left(1 - \frac{\ln(R+1)}{R}\right) = \infty.$$

Hence, the volume of the solid obtained by rotating R about the y-axis is not finite.

100. Show that $\int_0^\infty x^n e^{-x^2}\, dx$ converges for all $n > 0$. *Hint:* First observe that $x^n e^{-x^2} < x^n e^{-x}$ for $x > 1$. Then show that $x^n e^{-x} < x^{-2}$ for x sufficiently large.

SOLUTION For $x > 1$, $x^2 > x$; hence $e^{x^2} > e^x$, and $0 < e^{-x^2} < e^{-x}$. Therefore, for $x > 1$ the following inequality holds:

$$x^{n+2} e^{-x^2} < x^{n+2} e^{-x}.$$

Now, using L'Hôpital's Rule $n+2$ times, we find

$$\lim_{x\to\infty} x^{n+2} e^{-x} = \lim_{x\to\infty} \frac{x^{n+2}}{e^x} = \lim_{x\to\infty} \frac{(n+2)x^{n+1}}{e^x} = \lim_{x\to\infty} \frac{(n+2)(n+1)x^n}{e^x}$$
$$= \cdots = \lim_{x\to\infty} \frac{(n+2)!}{e^x} = 0.$$

Therefore,

$$\lim_{x\to\infty} x^{n+2} e^{-x^2} = 0$$

by the Squeeze Theorem, and there exists a number $R > 1$ such that, for all $x > R$:

$$x^{n+2} e^{-x^2} < 1 \quad \text{or} \quad x^n e^{-x^2} < x^{-2}.$$

Finally, write

$$\int_0^\infty x^n e^{-x^2}\, dx = \int_0^R x^n e^{-x^2}\, dx + \int_R^\infty x^n e^{-x^2}\, dx.$$

The first integral on the right-hand side has finite value since the integrand is a continuous function. The second integral converges since on the interval of integration, $x^n e^{-x^2} < x^{-2}$ and we know that $\int_R^\infty x^{-2}\, dx = \int_R^\infty \frac{dx}{x^2}$ converges. We conclude that the integral $\int_0^\infty x^n e^{-x^2}\, dx$ converges.