

# Constructions of multiplicative-periodic functions on $\mathbb{C}^*$

IOANA MIHAILA\*

Department of Mathematics, California State Polytechnic  
University Pomona, Pomona, CA 91768, USA

Communicated by J. Ryan

(Received in final form 5 March 2005)

This article studies the Riemann surfaces generated by factorization of the multiplicative group of nonzero complex numbers through its discrete subgroups. The first section of the article provides a classification of the discrete subgroups of  $(\mathbb{C}^*, \cdot)$ , and the Riemann surfaces that they generate. Meromorphic functions on these surfaces correspond to functions that are multiplicative-periodic in the complex plane and have essential singularities at 0 and  $\infty$ . The second part of the article presents constructions of multiplicative-periodic functions. The final section discusses the relationship between the functions constructed and elliptic double periodic functions.

*Keywords:* Riemann surfaces; Meromorphic functions; Discrete subgroup

*2000 Mathematics Subject Classifications:* 30D30; 30F99; 20F38

## 1. Introduction

The following example illustrates what the author is attempting to discuss in this article. When  $(\mathbb{C}, +)$  is considered as an additive topological group, its discrete subgroups can be generated by one or two elements. A subgroup generated by one element  $\alpha$ , is of the type  $\{m\alpha: m \in \mathbb{Z}\}$ . The quotient Riemann surface arising from factoring  $\mathbb{C}$  through the corresponding subgroup of translations is a cylinder. The meromorphic functions on this surface extend to meromorphic functions on  $\mathbb{C}$ , of period  $\alpha$ . A subgroup generated by two elements  $\omega_1$  and  $\omega_2$ , is of the type  $\{m\omega_1 + n\omega_2: m, n \in \mathbb{Z}\}$ , with  $\omega_1/\omega_2 \notin \mathbb{R}$ . The quotient Riemann surface arising from factoring  $\mathbb{C}$  through the corresponding subgroup of translations is a torus.

---

\*Email: imihaila@csupomona.edu

The meromorphic functions on such a torus extend to meromorphic functions on  $\mathbb{C}$ , which are double periodic of periods  $\omega_1$  and  $\omega_2$ .

Throughout this article, the set of complex numbers without zero will be denoted by  $\mathbb{C}^*$ .

Our purpose is to perform the analogous analysis in the case when  $\mathbb{C}^*$  is regarded as a multiplicative group. The validity of this construction is insured by the following theorem:

**THEOREM** *Let  $D$  be a subdomain of  $\mathbb{C} \cup \{\infty\}$  and let  $G$  be a group of Möbius transformations which leaves  $D$  invariant and which acts discontinuously in  $D$ . Then  $D/G$  is a Riemann surface [1].*

## 2. Discrete subgroups of $(\mathbb{C}^*, \cdot)$

Our first task is to find all the discrete subgroups of  $\mathbb{C}^*$ , considered with multiplication. These subgroups will be generated by one or two elements.

### A. Subgroups generated by one element

Let  $a = re^{i\theta}$ . The subgroup generated by  $a$  is  $\langle a \rangle = \{r^n e^{in\theta} : n \in \mathbb{Z}\}$ . This corresponds to the group of Möbius transformations  $G = \langle g(z) = az \rangle$ , acting on  $\mathbb{C}^*$ .

- (a) The case  $\theta = 0$ . In this case  $a = r \in \mathbb{R}_+$ ,  $a \neq 1$ . The orbit  $G(z)$  of a point  $z$  in  $\mathbb{C}^*$  is  $\{a^n z : n \in \mathbb{Z}\}$  and consists of points situated on a line through the origin. A fundamental domain is an annulus centred at the origin with ratio of the radii equal to  $a$  (see figure 1).
- (b) The case  $r = 1$ ,  $\theta \in 2\pi\mathbb{Q}$ . If  $\theta = (2\pi k)/n$ , with  $k$  and  $n$  relatively prime,  $\langle a \rangle = U_n$ , the group of  $n^{\text{th}}$  roots of 1. Therefore  $z_1 z_2^{-1} \in \langle a \rangle \Leftrightarrow (z_1/z_2)^n = 1 \Leftrightarrow r_1 = r_2$ ,  $\theta_1 - \theta_2 = (2\pi l)/n$ . The fundamental domain is an infinite sector of angle  $2\pi/n$ , centred at the origin (see figure 2).
- (c) The case  $r = 1$ ,  $\theta \notin 2\pi\mathbb{Q}$ . The group generated by  $a$  is not discrete in this case. One way to justify this is to notice that on the unit circle there are infinitely many points equivalent to 1, therefore they have an accumulation point.
- (d) The case  $r \neq 1$ ,  $\theta \in 2\pi\mathbb{Q}$ . Consider the smallest positive integer  $n$  such that  $a^n \in \mathbb{R}_+$ . For two nonzero complex numbers  $z_1$  and  $z_2$  to be equivalent,

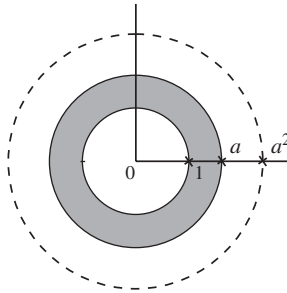


Figure 1. Case A.a.

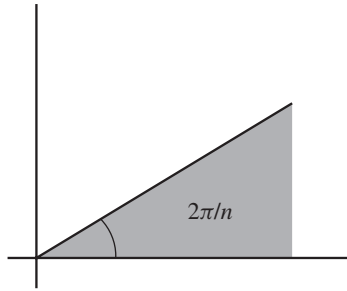


Figure 2. Case *A.b.*

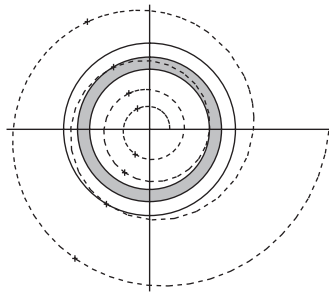


Figure 3. Case *A.d.*

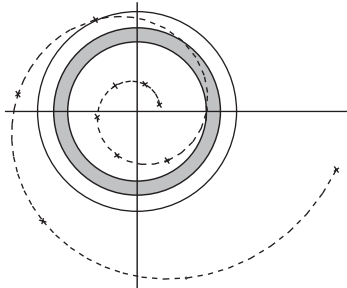


Figure 4. Case *A.e.*

we have  $z_1 z_2^{-1} \in \langle a \rangle \Leftrightarrow (z_1/z_2)^n = r^m \Leftrightarrow r_1 = r_2 r^m$ ,  $\theta_1 - \theta_2 = (2\pi k)/n$ . The points in the orbit  $G(z)$  of a given  $z$  are found at the intersection of a logarithmic spiral with the rays forming angles of  $(2\pi k)/n$  with the  $x$ -axis. The spiral is obtained considering the exponential transformation of the line that contains the points  $\{k(\ln r + i\theta): k \in \mathbb{Z}\}$ . As fundamental domains we can consider annuli with ratio of the radii equal to  $r$ . For example, if  $n = 3$ , they are pictured in figure 3.

- (e) The case  $r \neq 1$ ,  $\theta \notin 2\pi\mathbb{Q}$  is similar to the previous case, except the points in the orbit of  $z$  are not aligned on rays from the origin (see figure 4).

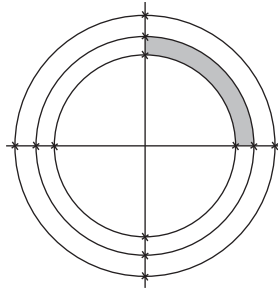


Figure 5. Case B.

### B. Subgroups generated by more than one element

Let  $a_1 = r_1 e^{i\theta_1}$  and  $a_2 = r_2 e^{i\theta_2}$ . We can assume that  $r_1, r_2 \geq 1$ . The only case that produces a discrete subgroup is the case when  $r_1 \neq 1$ , and  $r_1^m = r_2^l$ , for some  $m, l$  integers, not both equal to zero (consider the smallest nonnegative ones with this property), and  $\theta_1, \theta_2 \in 2\pi\mathbb{Q}$ . Then there exists  $r > 1$  such that  $r_1 = r^l$  and  $r_2 = r^m$ , and there exists  $\theta = (2\pi\kappa)/n$  such that  $\theta_i = k_i\theta$  for  $i=1, 2$  (consider the smallest positive  $\theta$  with this property and  $\kappa, n$  relatively prime). The subgroup generated by  $r_1, r_2$  is then equal to  $\langle r, e^{(2\pi i)/n} \rangle$ . The fundamental domains are quadrilaterals. An example with  $\theta = \pi/2$  is shown in figure 5.

### 3. Meromorphic functions on the quotient surfaces

The next goal is to construct meromorphic functions on the Riemann surfaces obtained via these factorizations. The extensions of these functions to  $\mathbb{C}^*$  via the quotient map  $\pi$  are multiplicative-periodic functions satisfying  $g(az) = g(z)$ , for every element  $a$  of the discrete group  $G$  (see figure 6).

*Case A.b.* Functions of the type  $g(z) = F(z^n)$ , where  $F$  is meromorphic, have multiplicative period  $e^{(2\pi i)/n}$ . This is the only case when the fundamental domain is equivalent to an infinite cylinder, or  $\mathbb{C}^*$ .

*Cases A.a., A.d., and A.e.* The fundamental domains are equivalent to tori. Meromorphic functions on these surfaces can be explicitly constructed through the following procedure.

Let  $z_0 \in \mathbb{C}^*$  and  $a = re^{i\theta}$  as above. Without loss of generality we can assume that  $r > 1$ . Weierstrass' theorem guarantees that the function  $f_{1,z_0}(z) = \prod_{n=0}^{\infty} (1 - \frac{z}{z_0 a^n})$

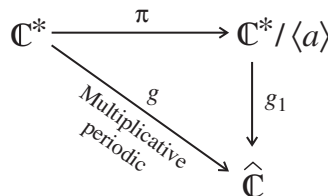


Figure 6. Extension of functions meromorphic on quotient surface to multiplicative-periodic functions.

is analytic on  $\mathbb{C}$  and has zeros  $a^n z_0$ ;  $n \geq 0$ . Using the transformation  $Z = 1/z$  we obtain the function  $f_{2, z_0}$ , analytic on  $\mathbb{C}^* \cup \{\infty\}$ , and with zeros at  $\{z_0/a^n$ ;  $n \geq 1\}$ , given by  $f_{2, z_0}(z) = \prod_{n=1}^{\infty} (1 - \frac{z_0}{z a^n})$ . Hence  $f_{z_0} = f_{1, z_0} \cdot f_{2, z_0}$  is analytic on  $\mathbb{C}^*$  and has zeros at the points of  $G(z_0) = \{a^n z_0$ ;  $n \in \mathbb{Z}\}$ . Note that

$$\begin{aligned} f_{z_0}(az) &= \prod_{n=0}^{\infty} \left(1 - \frac{z}{z_0 a^{n-1}}\right) \prod_{n=1}^{\infty} \left(1 - \frac{z_0}{a^{n+1} z}\right) \\ &= f_{z_0}(z) \frac{\left(1 - \frac{z a}{z_0}\right)}{\left(1 - \frac{z_0}{a z}\right)} = -\frac{a z}{z_0} f_{z_0}(z) \end{aligned}$$

Consider another point  $z_1 \in \mathbb{C}^*$  and repeat the construction above to obtain the function  $f_{z_1}$ , analytic on  $\mathbb{C}^*$ , which satisfies  $f_{z_1}(az) = -(az/z_1)f_{z_1}(z)$ . The function  $h = f_{z_0}/f_{z_1}$  is meromorphic on  $\mathbb{C}^*$ , with simple zeros at the points of  $G(z_0)$ , and simple poles at the points of  $G(z_1)$ . If we denote  $z_1/z_0$  by  $c$ , this function satisfies  $h(az) = c \cdot h(z)$ .

For all the groups in case *A*. with annuli as fundamental domains, choose  $z_1 = -z_0$ , and therefore  $h^2(az) = h^2(z)$ , making  $g = h^2$  the desired multiplicative-periodic function on  $\mathbb{C}^*$ , with double zeros at  $G(z_0)$  and double poles at  $G(z_1)$ .

For case *B.*, choose  $z_0 = 1$  and  $z_1 = e^{i\theta/2}$ , and construct  $h$  as above with zeros at  $\{r^k$ ;  $k \in \mathbb{Z}\}$  and poles at  $\{r^k e^{\pi i/n}$ ;  $k \in \mathbb{Z}\}$ . The function defined by  $g(z) = h^{2n}(z^n)$  is multiplicative-periodic with periods  $r^k e^{(2m\pi i)/n}$ .

#### 4. Multiplicative-periodic and double-periodic functions

Consider a double-periodic function  $f$  on  $\mathbb{C}$ , with periods  $\omega$  and  $\omega'$ , such that  $\omega$  and  $\omega'$  are linearly independent, and the transformation  $z = e^{(2\pi i w)/\omega}$ . The function  $\varphi$  defined by  $\varphi(z) = f(w)$  has no essential singularities except 0 and  $\infty$  and satisfies  $\varphi(\alpha z) = \varphi(z)$ , where  $\alpha = e^{(2\pi i \omega')/\omega}$  [3].

It is easy to establish an explicit connection between the meromorphic functions on the tori built above and the elliptic functions.

For cases *Aa.*, *Ad.*, and *Ae.*, choose as fundamental domains the annulus of radii 1 and  $a=r$ . The transformation  $w = \ln z$  maps the annulus into the parallelogram with sides  $\ln r + i\theta$  and  $2\pi i$  (see figure 7).

The points  $z_0 = 1$ ,  $z_1 = -1$ , are mapped to  $w_0 = 0$ ,  $w_1 = \pi i$ . Let

$$\sigma(z) = z \prod_{m, n=-\infty}^{\infty} \left(1 - \frac{z}{m(\ln r + \theta i) + i2\pi n}\right) e^{\frac{z}{m(\ln r + \theta i) + i2\pi n} + \frac{z^2}{2(m(\ln r + \theta i) + i2\pi n)^2}},$$

be the Weierstrass  $\sigma$ -function that has zeros at  $m(\ln r + \theta i) + n2\pi i$ . Therefore the function defined by  $f(w) := [\sigma(w)/\sigma(w - \pi i)]^2$  is a meromorphic double-periodic function that has double zeros at  $m \ln r + m\theta i + n2\pi i$  and double poles at  $m \ln r + m\theta i + (2n + 1)\pi i$ . Hence  $g(e^w) = f(w) \cdot C$ , for some constant  $C$ .

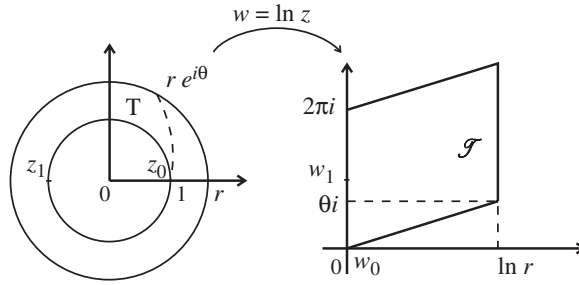


Figure 7. Fundamental domain and its image via the transformation  $w = \ln z$ .

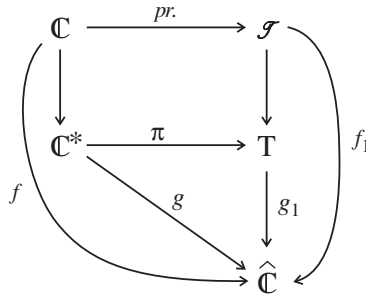


Figure 8. Relationship between multiplicative periodic functions, elliptic functions, and corresponding functions of the tori.

For case *B*, the construction is similar, except for the fact that we start with the fundamental domain  $\{z = se^{i\alpha}: 1 \leq s < r, 0 \leq \alpha < 2\pi/n\}$ , and we map it to the parallelogram with sides  $\ln r + i\theta$  and  $2\pi i/n$ .

The relationship between the multiplicative periodic function  $g$  constructed in the previous section, the elliptic function  $f$ , and the corresponding functions  $f_1$  and  $g_1$  defined on the tori is illustrated by the diagram in figure 8.

### 5. Questions and comments

The connection between multiplicative-periodic functions and double-periodic functions is well-known [3], even though not extensively used. However, the direct construction of multiplicative-periodic functions using the multiplicative structure on  $\mathbb{C}^*$  has not been explored. An interesting use of multiplicative-periodic functions in generating art was presented in [4].

The construction presented in this article relies on the clever choice of points in the fundamental domain. Many such choices are possible. It would be interesting to find in what measure this type of construction generates all the multiplicative periodic functions on  $\mathbb{C}^*$ .

## References

- [1] Beardon, A.F., 1983, *The Geometry of Discrete Groups* (New York: Springer-Verlag).
- [2] Farkas, H.M. and Kra, I., 1980, *Riemann Surfaces* (New York: Springer-Verlag).
- [3] Osgood, W.F., 1907, *Lehrbuch der Funktionentheorie* (Leipzig und Berlin: B.G. Teubner).
- [4] De Smit, B. and Lenstra, H.W. Jr, 2003, *Artful Mathematics: The Heritage of M. Escher*, *AMS Notices*, **50**, 446.