Midterm

CS 541
Fall 2005
Craig A. Rich

1 **Theorem**. $(D, \sqsubseteq_D), (E, \sqsubseteq_E)$ are CPOs $\Longrightarrow$ $(D \otimes E, \sqsubseteq)$ is complete; i.e.,

$$\forall \text{directed } M \subseteq D \otimes E \; \exists x \in D \otimes E \; [x = \sqcup M].$$

2 Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ be the set of natural numbers and $\leqslant$ be the binary relation on $\mathbf{N}$ defined as usual.

a) **Theorem**. $(\mathbf{N}, \leqslant)$ is a partially-ordered set.

b) **Theorem**. $(\mathbf{N}, \leqslant)$ is not complete; i.e.,

$$\neg \forall \text{directed } M \subseteq \mathbf{N} \; \exists x \in \mathbf{N} \; [x = \sqcup M].$$

3 Let $(\mathbf{N}_\perp, \sqsubseteq)$ be the flat CPO of natural numbers and $\text{succ}\colon \mathbf{N}_\perp \to \mathbf{N}_\perp$ be defined by

$$\text{succ}(x) = \begin{cases} \perp & \text{if } x = \perp \\ x + 1 & \text{if } x \in \mathbf{N}. \end{cases}$$

Theorem. succ is continuous.

4 Let $\mathbf{N}_\infty = \{0, 1, 2, \ldots\} \uplus \{\infty\}$, $\leqslant_\infty$ be the binary relation on $\mathbf{N}_\infty$ defined by

$$x \leqslant_\infty y \quad \text{if } (x, y \in \mathbf{N} \text{ and } x \leqslant y) \text{ or } y = \infty,$$

and $\text{succ}\colon \mathbf{N}_\infty \to \mathbf{N}_\infty$ be defined by

$$\text{succ}(x) = \begin{cases} x + 1 & \text{if } x \in \mathbf{N} \\ \infty & \text{if } x = \infty. \end{cases}$$

a) **Theorem**. $(\mathbf{N}_\infty, \leqslant_\infty)$ is a CPO.

b) **Theorem**. succ is continuous.

c) **Theorem**. $\text{fix}(\text{succ}) = \infty$.

5 Let $n \geqslant 2$, $(D_1, \sqsubseteq_1), (D_2, \sqsubseteq_2), \ldots, (D_n, \sqsubseteq_n)$ be CPOs, and $\text{on}_i\colon D_1 \times D_2 \times \cdots \times D_n \to D_i$ be defined by

$$\text{on}_i(x_1, x_2, \ldots, x_n) = x_i \quad \text{for } 1 \leq i \leq n.$$

Theorem. on_i is strict and continuous.

6 Consider the following denotational semantics specification:

- Syntax

$$
\begin{array}{ll}
\mathsf{P} \longrightarrow \varepsilon \mid \mathsf{P}_1\mathsf{S} & \text{(Path)} \\
\mathsf{S} \longrightarrow (\mathsf{L}_1, \mathsf{L}_2) & \text{(Step)} \\
\mathsf{L} \longrightarrow \text{“unspecified”} & \text{(Literal)}
\end{array}
$$

- Semantic Elements

  sum: $\mathbf{R}_\perp \otimes \mathbf{R}_\perp \circ\!\!\longrightarrow \mathbf{R}_\perp$,
  prod: $\mathbf{R}_\perp \otimes \mathbf{R}_\perp \circ\!\!\longrightarrow \mathbf{R}_\perp$,
  cos: $\mathbf{R}_\perp \circ\!\!\longrightarrow \mathbf{R}_\perp$,
  sin: $\mathbf{R}_\perp \circ\!\!\longrightarrow \mathbf{R}_\perp$ are defined as usual.

- Semantic Functions

  $\mathcal{L}$: Literal $\rightarrow \mathbf{R}_\perp$ is “unspecified”.

  $\mathcal{S}$: Step $\rightarrow \mathbf{R}_\perp \otimes \mathbf{R}_\perp$ is defined by
  $$\mathcal{S}[(\mathsf{L}_1, \mathsf{L}_2)] = \text{smash}(\mathcal{L}[\mathsf{L}_1], \mathcal{L}[\mathsf{L}_2])$$

  $\mathcal{P}$: Path $\rightarrow \mathbf{R}_\perp \otimes \mathbf{R}_\perp$ is defined by
  $$\mathcal{P}[\varepsilon] = (0, 0)$$
  $$\begin{aligned}\mathcal{P}[\mathsf{P}_1\mathsf{S}] = \text{smash}\Big(&\text{sum smash}\big(\text{on}_1\, \mathcal{P}[\mathsf{P}_1], \text{prod smash}(\text{on}_1\, \mathcal{S}[\mathsf{S}], \text{cos on}_2\, \mathcal{S}[\mathsf{S}])\big), \\ &\text{sum smash}\big(\text{on}_2\, \mathcal{P}[\mathsf{P}_1], \text{prod smash}(\text{on}_1\, \mathcal{S}[\mathsf{S}], \text{sin on}_2\, \mathcal{S}[\mathsf{S}])\big)\Big)\end{aligned}$$

Syntactically, a path P is a list of zero or more steps S, which are parenthesized comma-separated pairs of literals. Semantically, $\mathcal{S}[\mathsf{S}]$ and $\mathcal{P}[\mathsf{P}]$ are ordered pairs (or $\perp$) in $\mathbf{R}_\perp \otimes \mathbf{R}_\perp$.

a) What is the intuitive meaning of a step S in which $\mathcal{S}[\mathsf{S}] = \perp$?

b) What is the intuitive meaning of a path P in which $\mathcal{P}[\mathsf{P}] = \perp$?

c) What is the intuitive meaning of a step S in which $\mathcal{S}[\mathsf{S}] \neq \perp$?
Hint: consider how $\mathcal{S}[\mathsf{S}]$ is used in defining $\mathcal{P}$.

d) What is the intuitive meaning of a path P in which $\mathcal{P}[\mathsf{P}] \neq \perp$?
Hint: recall that $r \cos\theta$ and $r \sin\theta$ are the horizontal and vertical components of a vector of length r facing direction θ.