Midterm

CS 541
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1 Let $(D, \sqsubseteq)$ be a poset. Prove that least upper bound is unique; i.e., prove the following theorem.

Theorem. $\forall M \subseteq D,\ \forall x, y \in D,\ x = \sqcup M \wedge y = \sqcup M \implies x = y$.

In problems 2–3, let $\mathbf{N}^+ = \{1, 2, 3, \ldots\}$, and $\sqsubseteq$ be the binary relation on $\mathbf{N}^+$ defined by $x \sqsubseteq y$ if x divides evenly into y (with no remainder). For example, $3 \sqsubseteq 12$ and $3 \not\sqsubseteq 10$. Prove the following theorems.

2 **Theorem**. $(\mathbf{N}^+, \sqsubseteq)$ is a CPO.

3 Let $f\colon \mathbf{N}^+ \to \mathbf{N}^+$ be defined by

$$f(x) = \begin{cases} x, & \text{if } x \leq 100 \\ 2x, & \text{if } x > 100. \end{cases}$$

Theorem. f is monotone $\wedge$ f is not continuous.

4 Consider the following context-free grammar:

- Syntax

$$\begin{array}{lcll} G & \longrightarrow & L \mid L\Delta & \text{(Grade)} \\ L & \longrightarrow & \mathtt{A} \mid \mathtt{B} \mid \mathtt{C} \mid \mathtt{D} \mid \mathtt{F} & \text{(Letter)} \\ \Delta & \longrightarrow & \mathtt{+} \mid \mathtt{-} & \text{(Plus-Minus)} \end{array}$$

Design a denotational semantics specification in which $\mathbf{R}_\perp$ is the flat CPO of real numbers, and $\mathcal{G}\colon \text{Grade} \to \mathbf{R}_\perp$ is defined such that

$$\mathcal{G}[G] = \begin{cases} 4.0, & \text{if } G \text{ is } \mathtt{A} \\ 3.7, & \text{if } G \text{ is } \mathtt{A-} \\ 3.3, & \text{if } G \text{ is } \mathtt{B+} \\ 3.0, & \text{if } G \text{ is } \mathtt{B} \\ 2.7, & \text{if } G \text{ is } \mathtt{B-} \\ 2.3, & \text{if } G \text{ is } \mathtt{C+} \\ 2.0, & \text{if } G \text{ is } \mathtt{C} \\ 1.7, & \text{if } G \text{ is } \mathtt{C-} \\ 1.3, & \text{if } G \text{ is } \mathtt{D+} \\ 1.0, & \text{if } G \text{ is } \mathtt{D} \\ 0.7, & \text{if } G \text{ is } \mathtt{D-} \\ 0.0, & \text{if } G \text{ is } \mathtt{F} \\ \perp, & \text{otherwise.} \end{cases}$$

$\boxed{5}$ $(D, \sqsubseteq)$ is a *lattice* if

$$\begin{aligned} & 1)\ (D, \sqsubseteq) \text{ is a partially-ordered set} \\ \wedge\ & 2)\ \forall M \subseteq D,\ \exists x \in D,\ x = \sqcup M \\ \wedge\ & 3)\ \exists \text{least element } \bot_D \in D. \end{aligned}$$

Prove the following theorems:

a) **Theorem.** $\forall (D, \sqsubseteq),\ (D, \sqsubseteq)$ is a lattice $\Longrightarrow (D, \sqsubseteq)$ is a CPO.

b) **Theorem.** $\exists (D, \sqsubseteq),\ (D, \sqsubseteq)$ is a CPO $\wedge\ (D, \sqsubseteq)$ is not a lattice.

$\boxed{6}$ Consider the following denotational semantics specification:

- Syntax

$$\begin{array}{lcll} R & \longrightarrow & L & \text{(Roman-Numeral)} \\ L & \longrightarrow & \varepsilon \mid LT & \text{(List)} \\ T & \longrightarrow & D \mid D_1 D_2 & \text{(Term)} \\ D & \longrightarrow & \mathtt{I} \mid \mathtt{V} \mid \mathtt{X} \mid \mathtt{C} \mid \mathtt{M} & \text{(Digit)} \end{array}$$

- Semantic Functions

  $\mathcal{R}\colon \text{Roman-Numeral} \to \mathbf{N}_\bot$
  $\mathcal{L}\colon \text{List} \to \mathbf{N}_\bot \times \mathbf{N}_\bot$
  $\mathcal{T}\colon \text{Term} \to \mathbf{N}_\bot$
  $\mathcal{D}\colon \text{Digit} \to \mathbf{N}_\bot$

  are defined by

  $\mathcal{R}[L] = \text{on}_1(\mathcal{L}[L])$
  $\mathcal{L}[\varepsilon] = (0, 1000)$
  $\mathcal{L}[LT] = \big(\text{if} <\!(\text{smash}(\text{on}_2(\mathcal{L}[L]), \mathcal{T}[T]))\ \text{then}\ \bot\ \text{else sum}(\text{on}_1(\mathcal{L}[L]), \mathcal{T}[T]),\ \mathcal{T}[T]\,\big)$
  $\mathcal{T}[D] = \mathcal{D}[D]$
  $\mathcal{T}[D_1 D_2] = \text{if} <\!(\text{smash}(\mathcal{D}[D_1], \mathcal{D}[D_2]))\ \text{then diff}(\text{smash}(\mathcal{D}[D_2], \mathcal{D}[D_1]))\ \text{else}\ \bot$
  $\mathcal{D}[\mathtt{I}] = 1$
  $\mathcal{D}[\mathtt{V}] = 5$
  $\mathcal{D}[\mathtt{X}] = 10$
  $\mathcal{D}[\mathtt{C}] = 100$
  $\mathcal{D}[\mathtt{M}] = 1000$

a) Show that the context-free grammar is ambiguous, by giving two distinct derivation trees for the string of terminals `MCM`.

b) Compute $\mathcal{R}[\mathtt{MCM}]$ separately for each of the derivation trees in a).