Midterm

CS 531
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In the following problems, $\mathbf{N} = \{0, 1, 2, \ldots\}$ is the set of natural numbers, and $\mathbf{R}^*$ is the set of nonnegative real numbers.

1 Consider the following decision problem Π:

Knapsack

Instance: A finite set U,
a "size" $s(u) \in \mathbf{N}$ for each $u \in U$,
a "value" $v(u) \in \mathbf{N}$, for each $u \in U$,
positive integers B and K.

Question: Is there a subset $U' \subseteq U$ such that $\sum_{u \in U'} s(u) \leqslant B$ and $\sum_{u \in U'} v(u) \geqslant K$?

a) Give an encoding scheme $e \colon D_\Pi \to \Sigma^*$ (using an appropriate underlying alphabet Σ), by showing the encoding $e(I)$ of the following instance I:

u	$s(u)$	$v(u)$
0	19	12
1	6	4
2	11	14

, $B = 26, K = 17$.

b) Give a lower and upper bound on the length of an encoded instance $e(I)$, in terms of the components U, s, v, B, K of an instance I.

c) Give a simple "reasonable" length function $\text{length} \colon D_\Pi \to \mathbf{N}$ that is polynomially-related to the encoding function e.

2 Let $F = \{2, 3, 4, 5, 6, 7, 8, 9, 10, \mathtt{J}, \mathtt{Q}, \mathtt{K}, \mathtt{A}\}$,
$S = \{\diamondsuit, \clubsuit, \heartsuit, \spadesuit\}$,
$\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \ldots, \mathtt{z}, \mathtt{A}, \mathtt{B}, \mathtt{C}, \ldots, \mathtt{Z}\}$.

For each of the following kinds of instances I, give a simple "reasonable" length function $\text{length}(I)$:

a) Instance: $I \in (F \times S) \times (F \times S) \times (F \times S)$

b) Instance: $I \in (F \times \Sigma^*) \times (F \times \Sigma^*) \times (F \times \Sigma^*)$

c) Instance: $I \in \wp(\mathbf{N})$

d) Instance: $I \in \wp(F \times S)$

3 Let $f, g\colon \mathbf{N} \to \mathbf{R}^*$. In asymptotic running time/space analysis, we define

$$\begin{array}{ll} f \text{ is } O(g) & \text{if } \exists c_1{>}0\ \exists n_1{\geqslant}0\ \forall n{\geqslant}n_1\ [f(n) \leqslant c_1 g(n)], \\ f \text{ is } \Omega(g) & \text{if } \exists c_2{>}0\ \exists n_2{\geqslant}0\ \forall n{\geqslant}n_2\ [f(n) \geqslant c_2 g(n)], \\ f \text{ is } \Theta(g) & \text{if } f \text{ is } O(g) \text{ and } f \text{ is } \Omega(g). \end{array}$$

Prove the following theorem.

Theorem. f is $\Theta(g) \implies f$ is polynomially related to g.

4 Prove the following theorem.

Theorem. $\text{NTIME}(n) \subseteq \bigcup_{c \geqslant 1} \text{DSPACE}(2^{cn})$.

5 Prove the following theorem.

Theorem. $\text{DSPACE}(n/\ln n) \not\subseteq \text{DSPACE}(\ln^2 n)$.

6 Let $P = \{$'A', 'B', 'C'$\}$. In the following example, main() is a deterministic algorithm that invokes a recursive method move(**N**, P, P, P).

```
void move(n ∈ N, from ∈ P, to ∈ P, via ∈ P) {
    if (n > 0) {
        move(n−1, from, via, to);
        println("move from ", from, " to ", to);
        move(n−1, via, to, from);
    }
}

void main() {
    var n ∈ N;
    read(n);
    move(n, 'A', 'C', 'B');
}
```

What are the worst-case running time and the worst-case running space of main(), as functions of n?