

In problems 1–5, prove the given Theorems and Corollary. Recall that the diagonal language is defined by $L_d = \{ \langle M \rangle \mid \langle M \rangle \notin L(M) \}$.

1 The following Theorem shows that \mathcal{C} -hardness is closed upward under \leq_m :

Theorem. $\forall \mathcal{C} \subseteq \wp(\Sigma^*)$, L_1 is \mathcal{C} -hard and $L_1 \leq_m L_2 \implies L_2$ is \mathcal{C} -hard.

2 Let $L_\varepsilon = \{ \langle M \rangle \mid \varepsilon \in L(M) \}$.

a) **Theorem.** $L_\varepsilon \notin \text{REC}$.

b) **Theorem.** $L_\varepsilon \in \text{RE}$.

3 Let $L_\phi = \{ \langle M \rangle \mid L(M) = \phi \}$.

a) **Theorem.** $L_d \leq_m L_\phi$.

b) **Corollary.** $L_\phi \notin \text{RE}$.

4 The proof of Rice's Theorem actually shows that $\forall \wp, \phi \subset \wp \subset \text{RE} \implies L_\wp$ is RE-hard (Note: " \subset " denotes proper subset). The following Theorems show that for such a \wp , L_\wp may or may not be RE-complete:

a) **Theorem.** $\exists \wp, \phi \subset \wp \subset \text{RE}$ and L_\wp is RE-complete (Hint: use problem 2).

b) **Theorem.** $\exists \wp, \phi \subset \wp \subset \text{RE}$ and L_\wp is not RE-complete (Hint: use problem 3).

5 **Theorem.** $L_d \not\leq_m \overline{L_d}$ and $\overline{L_d} \not\leq_m L_d$.

6 In each of the following, answer true (T) or false (F):

a) \leq_m is transitive, reflexive, and symmetric.

b) \equiv_m is transitive, reflexive, and symmetric.

c) REC is closed under \equiv_m .

d) *EVENS*, *PRIMES*, L_u , $\overline{L_d}$, and *PCP* are languages in RE.

e) Rice's theorem implies that it is impossible to determine whether a given algorithm accepts exactly the primes.

f) The complement of a recursive language is recursively enumerable.

g) RE is closed under complement.

h) Every RE-complete language is not recursive.