Running Time Analysis

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Craig A. Rich

Expressing running time as a function $T : \mathbf{N} \to \mathbf{N}$

- *Iterative algorithms*

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad \text{if } 0 < c \neq 1.$$

- *Divide-and-conquer recursive algorithms*

$$T(n) = \begin{cases} 1, & \text{if } n \leq 1; \\ aT(n/b) + d(n), & \text{if } n > 1 \end{cases}$$

$$= \underbrace{n^{\log_b a}}_{\text{homogeneous solution}} + \underbrace{\sum_{j=0}^{k-1} a^j d(b^{k-j})}_{\text{particular solution}}, \quad \text{where } k = \log_b n,$$

$$\text{is } \begin{cases} \Theta(n^{\log_b a}), & \text{if } a > d(b); \\ \Theta(n^{\log_b d(b)}), & \text{if } a < d(b); \\ \Theta(n^{\log_b a} \log n), & \text{if } a = d(b), \end{cases} \quad \text{when } d(n) \text{ is multiplicative.}$$

Comparing functions $f, g : \mathbf{N} \to \mathbf{N}$

- *Definitions*

$$f(n) \text{ is } \mathrm{O}(g(n)) \iff \exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0, \quad f(n) \leq cg(n).$$

$$f(n) \text{ is } \underset{\text{a.e.}}{\Omega}(g(n)) \iff \exists c > 0 \; \exists n_0 > 0 \; \forall n \geq n_0, \quad f(n) \geq cg(n).$$

$$f(n) \text{ is } \Theta(g(n)) \iff \exists c_1, c_2 > 0 \; \exists n_0 > 0 \; \forall n \geq n_0, \quad c_1 g(n) \leq f(n) \leq c_2 g(n).$$

$$f(n) \text{ is } \underset{\text{i.o.}}{\Omega}(g(n)) \iff \exists c > 0 \; \forall n_0 > 0 \; \exists n \geq n_0, \quad f(n) \geq cg(n).$$

- *Theorems*

i) $f(n) \text{ is } \mathrm{O}(g(n)) \iff g(n) \text{ is } \underset{\text{a.e.}}{\Omega}(f(n)).$

ii) $f(n) \text{ is } \Theta(g(n)) \iff f(n) \text{ is } \mathrm{O}(g(n)) \text{ and } f(n) \text{ is } \underset{\text{a.e.}}{\Omega}(g(n)).$

iii) $f(n) \text{ is } \underset{\text{a.e.}}{\Omega}(g(n)) \implies f(n) \text{ is } \underset{\text{i.o.}}{\Omega}(g(n)).$

iv) $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \text{ is } \Theta(g(n)).$

v) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \implies f(n) \text{ is } \mathrm{O}(g(n)) \text{ and } f(n) \text{ is not } \underset{\text{a.e.}}{\Omega}(g(n)).$

vi) $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \text{ is } \underset{\text{a.e.}}{\Omega}(g(n)) \text{ and } f(n) \text{ is not } \mathrm{O}(g(n)).$

- *Differentiation rules useful in iv), v), and vi)*

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}.$$

$$\begin{aligned}
f(n) &= n^c &&\Longrightarrow f'(n) = cn^{c-1}.\\
f(n) &= \log_c n &&\Longrightarrow f'(n) = \frac{1}{n \ln c}.\\
f(n) &= c^n &&\Longrightarrow f'(n) = c^n \ln c.\\
f(n) &= g(n)h(n) &&\Longrightarrow f'(n) = g'(n)h(n) + g(n)h'(n).\\
f(n) &= g(h(n)) &&\Longrightarrow f'(n) = g'(h(n))h'(n).
\end{aligned}$$