Problem 25 Page 80

Suppose $\lim_{x \to x_0} f(x) = f(x_0)$. Note that $g(x) = \sup\{f(t) | a \leq t \leq x\}$ is increasing, and so by Lemma 2.7, g has a limit at x_0 iff

$$L(x_0) = \sup\{g(y) | y < x_0\} = g(x_0) = U(x_0) = \inf\{g(y) | x_0 < y\}.$$

We prove that g has a limit at x_0 by contradiction. So suppose $L(x_0) < U(x_0)$. We have $L(x_0) \leq g(x_0) \leq U(x_0)$, and so $L(x_0) < g(x_0)$ or (not and!) $g(x_0) < U(x_0)$. Now $f(x_0) \leq g(x_0)$ by definition of g. First suppose $f(x_0) < g(x_0)$. But $\lim_{x \to x_0} f(x) = f(x_0)$, so there is a small interval about x_0 on which $f(x) < g(x_0)$, and so g is constant on this interval(why?). But then g has a limit at x_0, contrary to our contradiction hypothesis. Thus $f(x_0) = g(x_0)$.

Now assume $L(x_0) < g(x_0)$. Let $\delta > 0$ be given. There is a y with $x_0 - \delta < y < x_0$ and $g(x_0) - g(y) \geq (g(x_0) - L(x_0))$. But there is a t with $x_0 - \delta < t < y$ such that $f(t) \leq g(y)$. Thus $f(x_0) - f(t) \geq (g(x_0) - L(x_0))$. Therefore, f has no limit at x_0.

Finally, assume $g(x_0) < U(x_0)$, and let $\delta > 0$ be given. We can fix any y with $x_0 < y < x_0 + \delta$. For that y, $g(x_0) < U(x_0) \leq g(y)$. So there must be a t such that $x_0 < t \leq y < x_0 + \delta$ and $f(t) > g(x_0) + \frac{U(x_0) - g(x_0)}{2}$(why?). So $f(t) - f(x_0) \geq \frac{U(x_0) - g(x_0)}{2}$, and hence f has no limit at x_0.