Article 1
Representations of the Poincaré Group

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1 Introduction

The purpose of these articles is to attempt to explain to myself and others certain materials from Steven Weinberg's *The Quantum Theory of Fields* by trying to cast these materials in more precise mathematical languge. The material of this article is the description of all the representations of the Poincaré or Inhomogeneous Lorentz group. The method we shall use is due to G. Mackey, and it is called the inducing construction. In brief, the inducing construction generates representations of a group from representations of a subgroup. We shall see that this method exhausts all the possibilities for representations of the Poincaré group. First, we describe the group.

2 The Poincaré Group

The Poincaré group is the group of transformations of R^4 produced by applying a Lorentz transformation Λ and then following it with a translation a in R^4. We can denote this as a pair (Λ, a). Given two such transformations (Λ_1, a_1) and (Λ_2, a_2), we can apply them in turn to $x \in R^4$ to get

$$\bar{x} = (\Lambda_2, a_2)(\Lambda_1, a_1)x = (\Lambda_2, a_2)(\Lambda_1 x + a_1) = \Lambda_2\Lambda_1 x + \Lambda_1 a_1 + a_2.$$

Thus

$$\bar{x} = (\Lambda_2\Lambda_1, \Lambda_1 a_1 + a_2)x,$$

and so this result is a transformation of the same type. The inverse of (Λ, a) is $(\Lambda^{-1}, -\Lambda^{-1}a)$, so we have a group.

The Poincaré group is a special example of a semi–direct product. A semi–direct product of two groups H and A is the set $H \times A$ together with the operation

$$(h_2, a_2)(h_1, a_1) = (h_2 h_1, a_2 \tau(h_2) a_1)$$

where $\tau : H \rightarrow Aut(A)$ is a fixed homomorphism of H into the automorphism group of A. In the case of the Poincaré group, τ is the embedding of the Lorentz group in the group of automorphisms of R^4.

To complete the description of the Poincaré group we need a brief description of the Lorentz group. A Lorentz transformation is a transformation that relates the coordinates of events as measured in two inertial frames O and $\bar{O}$. It is assumed that the origins of the frames coincide at the initial setting of both clocks to zero. The constancy of the speed of light in both frames leads to the conclusion that if (t, x, y, z) are the coordinates of an event as measured in O and $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ are the corresponding coordinates as measured in $\bar{O}$, then (c=1 in our units)

$$t^2 - x^2 - y^2 - z^2 = \bar{t}^2 - \bar{x}^2 - \bar{y}^2 - \bar{z}^2.$$

In other words, a Lorentz transformation is a linear transformation of R^4 that leaves the quadratic form $Q(u, v)$ with signature 1, -1, -1, -1 invariant. In terms of matrices, the condition is

$$\Lambda^t \eta \Lambda = \eta \; , (*)$$

where η is the diagonal matrix with entries 1, -1, -1, -1 down the main diagonal and Λ^t is the transpose of the Lorentz matrix Λ. We formally define the Lorentz group to be the group of 4×4 matrices satisfying this condition.

The relation $(*)$ immediately implies that $det(\Lambda) = |\Lambda| = 1$ in the connected component of the identity. In coordinates, we also have for $\Lambda = (\Lambda^\mu_\rho)$

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma}.$$

Whence

$$\eta_{00} = \eta_{\mu\nu} \Lambda^\mu_0 \Lambda^\nu_0 = (\Lambda^0_0)^2 - \sum_i \Lambda^i_0 \Lambda^i_0.$$

So $(\Lambda^0_0)^2 \geq 1$, and so $\Lambda^0_0 \geq 1$ or $\Lambda^0_0 \leq -1$. Also the length of $(\Lambda^1_0, \Lambda^2_0, \Lambda^3_0)$ is $\sqrt{(\Lambda^0_0)^2 - 1}$. By Schwartz inequality

$$|\bar{\Lambda}^0_i \Lambda^i_0| \leq \sqrt{(\Lambda^0_0)^2 - 1}\sqrt{(\bar{\Lambda}^0_0)^2 - 1}.$$

Now $(\bar{\Lambda}\Lambda)^0_0 = \bar{\Lambda}^0_\mu \Lambda^\mu_0$, Therefore, if $\Lambda^0_0 \geq 1$ and $\bar{\Lambda}^0_0 \geq 1$ then

$$(\bar{\Lambda}\Lambda)^0_0 \geq \bar{\Lambda}^0_0 \Lambda^0_0 - \sqrt{(\Lambda^0_0)^2 - 1}\sqrt{(\bar{\Lambda}^0_0)^2 - 1} \geq 1.$$

The conclusion is that the Lorentz transformations of determinant 1 and with $\Lambda^0_0 \geq 1$ form a subgroup of the Lorentz group. This subgroup is called the proper orthochronous group, and it is denoted by $L^\uparrow_+$. We shall take the Poincaré group to be the semi–direct product of $L^\uparrow_+$ with R^4.

We close this section by showing that frames related by a Lorentz transformation move with constant velocity with respect to one another and that the Lorentz transformation of motion along a coordinate axis has a simple form. Consider the motion of the origin of frame O as seen in frame $\bar{O}$. In $\bar{O}$ the space–time coordinates of $(t, 0, 0, 0)$ are

$$\bar{x}^\sigma = \Lambda^\sigma_\mu x^\mu = \Lambda^\sigma_0 t.$$

In particular,

$$\bar{t} = \Lambda^0_0 t,$$

and so

$$\frac{d\bar{x}^i}{d\bar{t}} = \frac{d\Lambda^i_0 t}{d\bar{t}} = \frac{d\bar{t}\Lambda^i_0/\Lambda^0_0}{d\bar{t}} = \frac{\Lambda^i_0}{\Lambda^0_0}.$$

Thus the speed is

$$\frac{\sqrt{(\Lambda^1_0)^2 + (\Lambda^2_0)^2 + (\Lambda^3_0)^2}}{\Lambda^0_0} = \frac{\sqrt{(\Lambda^0_0)^2 - 1}}{\Lambda^0_0}.$$

Note that if $\Lambda^0_0 = 1$, then the frames are at rest. Also note that the speed is always less than 1.

Of course, the coordinates in $\bar{O}$ can be rotated so that

$$\frac{d\bar{x}^2}{d\bar{t}} = \frac{d\bar{x}^3}{d\bar{t}} = 0,$$

in which case $\Lambda^2_0 = \Lambda^3_0 = 0$, and

$$\Lambda^1_0 = \sqrt{(\Lambda^0_0)^2 - 1}.$$

This means that $\bar{y}$ and $\bar{z}$ must be independent of t. The velocity in the $\bar{x}^1$ direction is

$$\beta = \frac{\sqrt{(\Lambda^0_0)^2 - 1}}{\Lambda^0_0}.$$

Then

$$\Lambda^0_0 = \frac{1}{\sqrt{1 - \beta^2}}.$$

The space coordinates of O can also be rotated so that the motion of the origin in $\bar{O}$ coordinates is along the x–axis in O coordinates. We assume both axes are rotated so that the motion is in the positive direction in both cases. Let us call the resulting Lorentz transformation $A = (a_{\mu\nu})$.

The inverse of A can be computed easily by observing that

$$A^t\eta = \eta A^{-1}.$$

The result is

$$A^{-1} = \begin{bmatrix} a_{00} & -a_{10} & -a_{20} & -a_{30} \\ -a_{01} & a_{11} & a_{21} & a_{31} \\ -a_{02} & a_{12} & a_{22} & a_{32} \\ -a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Because of the chosen orientation of the x and $\bar{x}$ axes, the form of the matrix reduces to

$$A = \begin{bmatrix} a_{00} & a_{01} & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ 0 & a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Using the fact that A leaves the quadratic form invariant leads, after some work, to the final form for A, namely

$$A = \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ \frac{\beta}{\sqrt{1-\beta^2}} & \frac{-1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3 Equivalent Representations and Irreducibility

In this section we quickly review the notions of equivalence and irreducibility of representations. Let G be a group and $U : G \to Aut(H)$ and $V : G \to Aut(K)$ representations. We say that a bounded operator $T : H \to K$ intertwines U and V if

$$TU_g = V_g T.$$

The set of intertwining operators from U to V is denoted by $R(U, V)$. If $R(U, V)$ contains a unitary operator we say U and V are equivalent.

Let $U : G \to Aut(H)$ be a representation and K a closed subspace of H. If U leaves $K \neq 0$ invariant then U restricted to K is also a representation of G. This is called a subrepresentation of U. We say U is irreducible if U has no proper subrepresentations.

The following is a generalization of Schur's Lemma. The operator details may be found in Kato.

Lemma 3.1 (Schur)
Let T belong to $R(U,V)$. Then U restricted to the orthogonal complement of the the kernel of T is equivalent to V restricted to the closure of the range of T.

Proof: It is immediate that the kernel and range of T are invariant under U and V respectivley. Thus the orthogonal complement of the kernel and the closure of the range are invariant also. Let $N^\perp$ be the orthogonal complement of the kernel and $\bar{R}$ the closure of the range. The polar decomposition of the operator T is

$$T = WA$$

where W is a partial isometry of $N^\perp$ onto $\bar{R}$ and A is non–negative and self–adjoint. Now T and A satisfy

$$\langle Tx, Ty\rangle = \langle Ax, Ay\rangle.$$

Then

$$\langle TU_g x, Ty\rangle = \langle AU_g x, Ay\rangle = \langle A^2 U_g x, y\rangle,$$

and

$$\langle TU_g x, Ty\rangle = \langle V_g Tx, Ty\rangle = \langle Tx, V_g^* Ty\rangle = \langle Tx, TU_g^* y\rangle$$
$$= \langle Ax, AU_g^* y\rangle = \langle U_g A^2 x, y\rangle.$$

Thus U_g commutes with A^2 and hence with A.

We have

$$WU_g Ax = WAU_g x = TU_g x = V_g Tx = V_g WAx,$$

so on the range of A which is $N^\perp$, W intertwines U and V. Therefore, U and V are equivalent on the orthogonal complement. •

Corollary 3.2
$R(U,V) = 0$ if and only if no subrepresentations of U and V are equivalent.

Proof: If $R(U,V) \neq 0$ then Schur's lemma gives equivalent subrepresentations. If there is a pair of equivalent subrepresentations then the unitary intertwining operator can be extended to a partial isometry intertwining U and V. •

Corollary 3.3
If U and V are irreducible then $R(U,V) = 0$ if and only if U and V are not equivalent.

Corollary 3.4
U is irreducible if and only if $R(U) = R(U,U)$ consists only of scalar multiples of the identity.

Proof: If U is irreducible then U is equivalent to itself,and so $R(U) \neq 0$. If T commutes with U then so does TT^*. Now if T is not a scalar multiple of the identity, then TT^* has a proper non–trivial projection. But this is impossible. $\bullet$

Finally note that if G is abelian and U is irreducible, then each U_g commutes with U, and so $U_g = \chi(g)I$. Clearly, $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$. Now I must be one dimensional, or the representation is reducible.

4 The Inducing Construction

We now describe a method of generating a representation of a group from a representation of a subgroup.

Definition 4.1
We say a group G acts on a space X if for each $g \in G$ there is a bijection τ_g of X such that τ_e is the identity, and $\tau_{gh} = \tau_g \tau_h$. We will usually write $\tau_g(x) = gx$. $\bullet$

The orbit of a point x_0 in X under the action of G is

$$Orb(x_0) = \{gx_0 | g \in G\},$$

and the stabilizer of x_0 is

$$Stab(x_0) = \{g | gx_0 = x_0\}.$$

If the orbit of a point is all of X we say the action is transitive and that X is a homogeneous G–space. In many cases, the stabilizer of a point is called a “little group”.

Example 4.2
The positive orthochronus Lorentz group $L_+^\uparrow$ acts on R^4. The orbits are the forward and backward light cones and the spacelike and timelike hyperboloids. For

example, if $x_0 = (m, 0, 0, 0)$, then $Orb(x_0)$ is the set of all points $p = \Lambda x_0$ with quadratic form value m^2. Thus the coordinates satisfy $t^2 - x^2 - y^2 - z^2 = m^2$. This is a hyperboloid of two sheets, but it is only the sheet containing x_0 that is the orbit. The spacelike hyperboloids are of one sheet. •

Theorem 4.3
If X is a homogeneous G–space, then $Stab(x_1)$ is isomorphic to $Stab(x_2)$.

Proof: Let $x_2 = gx_1$ and $h \in Stab(x_2)$. Then $g^{-1}hgx_1 = g^{-1}x_2 = x_1$, so $g^{-1}hg \in Stab(x_1)$. This is clearly and isomorphism. •

Definition 4.4
Let X_1 and X_2 be homogeneous G–spaces. We say X_1 is equivalent to X_2 if there is a bijection $\phi : X_1 \to X_2$ such that $\phi(gx) = g\phi(x)$ for all g and x. •

Theorem 4.5
Let X be a homogeneous G–space and x_0 a point in X. Let G_0 be the little group of x_0. Note that G acts transitively on the set of left cosets G/G_0. As G–spaces, X and G/G_0 are equivalent.

Proof: Define ϕ by $\phi(gG_0) = gx_0$. This is well–defined and a bijection. Also, $\phi(hgG_0) = hgx_0 = h\phi(gG_0)$. •

Examples 4.6
Consider the action of the proper orthochronus group $L_+^\uparrow$ on R^4 again.

a) If x_0 is forward or backward timelike, then $G_0 = SO(3)$. Indeed, we may take $x_0 = (m, 0, 0, 0)$, and so the little group elements must have $a_{00} = 1$, $a_{10} = 0$, $a_{20} = 0$, and $a_{30} = 0$. But from the formula for the inverse $a_{01} = 0$, $a_{02} = 0$, and $a_{03} = 0$, and so the elements of the little group are pure rotations. Thus the orbit, which is called a mass hyperboloid, is equivalent to $L_+^\uparrow/SO(3)$

b) If x_0 is on the light cone, then the little group is $ISO(2)$ the Euclidean 2–group. This is very much harder to show. We start with the universal cover $SL(2, C)$ of $L_+^\uparrow$.

Let τ be the map

$$\tau(x_0, x_1, x_2, x_3) = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_3 & x_0 - x_3 \end{bmatrix}.$$

If $x = (x_0, x_1, x_2, x_3)$, then for $A \in SL(2, C)$

$$\Lambda(A)(x) = \tau^{-1}(A\tau(x)A^*)$$

defines a linear transformation of R^4. Also, $Det(\tau(x)) = Q(x, x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$, and so

$$Q(\Lambda(A)(x), \Lambda(A)(x)) = Det(A\tau(x)A^*) = Q(x).$$

Thus $\Lambda(A)$ is a Lorentz transformation. It can be shown that this transformation doubly covers $L_+^\uparrow$. The invariance of a lightlike vector, say $(1, 0, 0, 1)$, is equivalent to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

From this and $ad - bc = 1$ it follows immediately that $|a|^2 = 1$ and $c = 0$. Therefore the most general member of $SL(2, C)$ that leaves $(1, 0, 0, 1)$ invariant is of the form

$$\begin{bmatrix} e^{-i\theta/2} & (\alpha + i\beta)e^{i\theta/2} \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

This matrix can be factored into

$$\begin{bmatrix} e^{-i\theta/2} & (\alpha + i\beta)e^{i\theta/2} \\ 0 & e^{i\theta/2} \end{bmatrix} = \begin{bmatrix} 1 & (\alpha + i\beta) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.$$

Let us call the first factor $T(\alpha, \beta)$ and the second $R(\theta)$. One easily checks that $T(\alpha, \beta)T(\bar{\alpha}, \bar{\beta}) = T(\alpha + \bar{\alpha}, \beta + \bar{\beta})$ and $R(\theta)R(\bar{\theta}) = R(\theta + \bar{\theta})$. A little effort, or a CAS, shows that $R(\theta)T(\alpha, \beta)R(-\theta) = T(\alpha\cos(\theta) + \beta\sin(\theta), -\alpha\sin(\theta) + \beta\cos(\theta)$. But these are the same relations satisfied by rotation and translation in the Euclidean group, so the little group fixing a lightlike vector is isomorphic to $ISO(2)$. If $(r, (\alpha, \beta))$ is in $ISO(2) = S^1 \bar{\times} R^2$ where the rotation r is rotation through and angle θ then the image element in G_0 under the isomorphism is $R(\theta)T(\alpha, \beta)$. •

Let X be a G–space. We say a measure μ on X is G–invariant if for each measurable set $E \subset X$ and each $g \in G$,

$$\mu(gE) = \mu(E).$$

Theorem 4.7 (Harr)
Let G act on itself by left multiplication. Then G has a left–invariant non–zero measure μ. This measure is unique up to a positive scalar multiple, and if G is compact, then μ is finite.

We now describe the inducing construction. Let X be a transitive G–space with a G–invariant measure μ. It can happen that there is no G–invariant measure on X to use. If so, then the day can be saved by the Radon–Nikodym theorem and what are called quasi–invariant measures. We will discuss this shortly. Now $X \equiv G/G_0$ where G_0 is a stability subgroup of x_0. So for each $x \in X$ we may choose $c(x)$ such that $c(x)x_0 = x$ and $c(x_0) = e$. Indeed, if x corresponds to the left coset gG_0, then we can set $c'(x)$ equal to any element in this coset. Then we can define $c(x) = (c'(x_0))^{-1}c'(x)$. This function can be chosen so that it is appropriately measurable, and it is called a Borel section. With this section, we can now induce a representation of G from a representation of G_0.

Let

$$L : G_0 \to Aut(K)$$

be a unitary representation of G_0 on the Hilbert space K. Now define a unitary operator U_g on the space of K–valued μ–measurable functions on X, $L^2(X, K, \mu)$, by

$$U_g(\phi)(x) = L(c(x)^{-1}gc(g^{-1}x))\phi(g^{-1}x).$$

The inner product is

$$\langle \phi, \psi \rangle = \int_X \langle \phi, \psi \rangle_K \, d\mu,$$

and so U_g is clearly unitary because of the G-invariance of μ. Also note that $c(x)^{-1}gc(g^{-1}x)x_0 = c(x)^{-1}gg^{-1}x = c(x)^{-1}x = x_0$, so $c(x)^{-1}gc(g^{-1}x)$ is in G_0.

The homomorphism law is verified as follows:

$$U_g(U_h\phi)(x) = L(c(x)^{-1}gc(h^{-1}x))(U_h\phi)(g^{-1}x)$$

$$= L(c(x)^{-1}gc(g^{-1}x)c(g^{-1}x)^{-1}hc(h^{-1}g^{-1}x))\phi(h^{-1}g^{-1}x)$$
$$= L(c(x)^{-1}ghc(h^{-1}g^{-1}x))\phi(h^{-1}g^{-1}x) = (U_{gh}\phi)(x).$$

The representation is strongly continuous, but the proof is beyond the scope of this outline.

We say that the representation U is induced by L, and we often write U^L for this induced representation. It is not hard to show that if L and $\bar{L}$ are unitarily equivalent, then so are U^L and $U^{\bar{L}}$. Moreover, up to unitary equivalence, it does not matter what Borel section c is chosen. The dependence of the induced representation on the measure, and the possible non-existence of a G–invariant measure, will take a few more words.

If G is compact then the quotient map

$$\pi : G \to G/G_0$$

allows us to define a G invariant measure on $X \equiv G/G_0$ as follows. Let $S \subset G/G_0$, and define $\mu_0(S) = \mu(\pi^{-1}S)$. Since G is compact this must be finite. However, if G is not compact, this construction may well fail. To recover from this, we need the notion of a quasi–G–invariant measure.

Let X be a homogeneous G–space, and μ a measure on X. We say μ is quasi–G–invariant if for all measurable $E \subset X$ and $g \in G$, $\mu(E) = 0$ implies that $\mu(g^{-1}E) = 0$. We say two measures are equivalent if they are absolutely continuous with respect to each other, i.e. they have the same null sets. This is an equivalence relation on the Borel measures on X. An equivalence class is called a measure class. Such a measure class is called G–invariant if the class is closed under the action $\mu \to \mu_g$ where $\mu_g(E) = \mu(g^{-1}E)$. Note that a class is G–invariant if and only if it contains a quasi–invariant measure. We can now state the following important theorem.

Theorem 4.8
Let G_0 be a closed subgroup of G and $X = G/G_0$. Then X has exactly one invariant measure class.

Example 4.9
Let $G = SO(3) \bar{\times} R^3$ be the Euclidean group on R^3 (a semi–direct product). Let G_0 fix the origin, so $G_0 = SO(3)$, and $G/SO(3) \equiv R^3$. Lebesgue measure belongs to the invariant measure class, and all other members of the class are absolutely continuous with respect to Lebesgue measure. For example, the measure defined by

$$\mu(E) = \int_E e^{-(x^2+y^2+z^2)}\, dx dy dz$$

is in this class. •

One may well ask if the inducing construction can still be carried out while using a measure from the unique invariant measure class. The answer is yes, but we need to use the Radon–Nikodym derivative. Let μ be chosen in the class, then μ_g defined by $\mu_g(S) = \mu(g^{-1}S)$ is mutually absolutely continuous with μ. Let r_g be the Radon–Nikodym derivative of μ with respect to $\mu_{g^{-1}}$, that is

$$\mu(S) = \int_S r_g(x) d\mu_{g^{-1}}.$$

Then we define

$$(U_g\phi)(x) = [r_g(g^{-1}x)]^{1/2} L(c(x)^{-1} g c(g^{-1}x))\phi(g^{-1}x).$$

It is not much more difficult to prove unitarity and the homomorphism law than it was before.

It can be shown that different measures from the invariant class give rise to equivalent induced representations. Moreover, the inducing construction preserves direct sums. That is

$$U^{L_1 \bigoplus L2} \equiv U^{L_1} \bigoplus U^{L_2}.$$

It is natural to ask if all representations of G can be induced from G_0. The answer is no because induced representations share a property that representations do not in general possess.

Let χ_E be the characteristic function of $E \subset X$. Then $P(E)$ defined by $(P(E)\phi)(x) = \chi_E(x)\phi(x)$ defines a projection valued measure on $L^2(X, K, \mu)$. For simplicity we assume μ is G–invariant, and the following shorthand notation will prove useful. Let $f(g, x) = L(c(x)^{-1} g c(g^{-1}x))$, then it is easy to see that $I = f(g, x) f(g^{-1}, g^{-1}x)$. Thus one readily computes that

$$(U_g P(E) U_g^{-1}\phi)(x) = f(g, x)\chi_E(g^{-1}x) f(g^{-1}, g^{-1}x)\phi(x)$$

$$= \chi_E(g^{-1}x)\phi(x) = (P(gE)\phi)(x).$$

Therefore, we have

$$U_g P(E) U_g^{-1} = P(gE).$$

The following definition is now called for.

Definition 4.10
Let $U : G \to Aut(H)$ be a representation of G. Let P be a projection valued measure on X with values in H such that $U_g P(E) U_g^{-1} = P(gE)$ for all $g \in G$ and $E \subset X$. We call the pair (U, P) a system of imprimitivity. We say a system of imprimitivity is irreducible if the only invariant subspaces of H under all U_g and $P(E)$ are 0 and H. •

Two systems of imprimitivity, (U, P) and $(\bar{U}, \bar{P})$ are unitarily equivalent if there is a unitary map $W : H \to \bar{H}$such that $\bar{U}_g W = W U_g$ and $P(\bar{E})W = W P(E)$ for all g and E.

Theorem 4.11 (Mackey)
Let X be a transitive G–space with G_0 the little group of x_0. Then the inducing construction gives a one to one correspondence between the unitary equivalence classes of systems of imprimitivity and the unitary equivalence classes of representations of G_0. Moreover, the correspondence preserves irreducibility and direct sums.

Example 4.12 (Localization)
Suppose that a particle has position coordinates. That is suppose that we have a projection valued measure P on R^3 such that for each state ϕ

$$\langle P(E)\phi, \phi\rangle$$

is the probability of measuring the particle in volume E in state ϕ. Let $G = SO(3)\bar{\times}R^3$ be the Euclidean group of motions of three dimensional space, and let $g \in G$ be the change of coordinates between a given frame and another. Then if E is a volume in the first frame then gE is the corresponding volume described in the second frame. Moreover, if U is the representation of G that relates states between Euclidean frames, then if ϕ is the state in the first frame, then $U_g(\phi)$ is the corresponding state in the second frame. Therefore,

$$\langle P(gE)U_g\phi, U_g\phi\rangle = \langle P(E)\phi, \phi\rangle,$$

and so

$$U_g P(E) U_g^{-1} = P(gE).$$

We say that particles that have such a system of imprimitivity are localizable.

Mackey's theorem tells us how to survey all irreducible systems of imprimitivity for localizable particles. Observe that if $G_0 = SO(3)$ then $G/G_0 \equiv R^3$, and the measure class contains Lebesgue measure which is invariant under the action of G. Therefore, if

$$L : SO(3) \rightarrow Aut(K)$$

is an irreducible representation then the corresponding irreducible system of imprimitivity is

$$U_g(\phi)(x) = L((I, x)^{-1} g(I, g^{-1}x))\phi(g^{-1}x)$$

and

$$(P(E)\phi)(x) = \chi_E \phi(x)$$

for $\phi \in L^2(R^3, K, \mu)$.

If $g = (R, a)$ then $(I, x)^{-1}g(I, g^{-1}x) = (I, x)^{-1}(R, a)(I, R^{-1}x - R^{-1}a) = (R, a - x)(I, R^{-1}(x - a)) = (R, 0)$. So the representation can be written more simply as

$$(U_g\phi)(x) = L(R)\phi(R(x) + a).$$

Thus we have a concrete representation of a localizable particle whose generators are the familiar momentum, angular momentum, and spin operators. •

We have not determined all the representations of the Euclidean group, only the systems of imprimitivity. However, we shall see that it is possible to describe all the representations of groups that have the same form as the Euclidean group, that is, groups of the form $H \bar{\times} A$ where A is abelian. Note that the Poincaré group is of this type. For this reason we take a brief detour into the representations of abelian groups.

5 Representations of Abelian Groups

If $U : G \to Aut(H)$ is an irreducible representation of an abelian group, then we have seen that $U(g)z = \chi(g)z$ where $\chi : G \to U(1) = S^1$ is a homomorphism into the circle group and $z \in C$. Irreducible abelian representations are one dimensional. Thus we may identify the irreducible representations with the characters χ. Describing all representations of an abelian group G will take more time.

Let $\widehat{G}$ be the set of all continuous homomorphisms $\hat{g} : G \to U(1)$. This is a sort of dual of the group G. One may ask if G is "reflexive". Note that $U^h : \widehat{G} \to U(1)$ defined by

$$U^h_{\hat{g}}(z) = \hat{g}(h)z$$

defines a character of $\widehat{G}$, thus $U^h \in \widehat{\widehat{G}}$. A theorem of Van–Kampen and Pontryagin says that every one dimensional representation of $\widehat{G}$ is if this form. Indeed, $\widehat{\widehat{G}}$ is topologically and algebraically isomorphic to G.

Examples 5.1
1) If χ is a character in $\widehat{R}$, then $\chi(r) = e^{ih(r)}$ where h is real, continuous, and additive. Thus $h(r) = pr$, and so $\chi(r) = e^{ipr}$. Thus $\widehat{R}$ is isomorphic to R. Note that this extends to R^n in the same way.

2) If χ is a character in $\widehat{U(1)}$, then $\chi(z) = e^{ih(arg(z))}$. Now h is continuous and linear on $0 \leq arg(z) < 2\pi$. So $h(r) = nr$ where n must be an integer. •

Now we consider other possible representations of an abelian group G. Let μ be a Borel measure on $\widehat{G}$ (keep R^4 in mind), and let $L^2(\widehat{G}, \mu)$ be the space of μ–square integrable functions on $\widehat{G}$. Then a representation $U_\mu : G \to Aut(L^2(\widehat{G}, \mu))$ can be defined by

$$(U_\mu(g)\phi)(\hat{g}) = \hat{g}(g)\phi(\hat{g}).$$

This is immediately seen to be unitary.

If μ_1 and μ_2 are equivalent measures, i.e. mutually absolutely continuous, then it is not hard to show that U_{μ_1} and U_{μ_2} are equivalent. Much harder to show is the so called S.N.A.G. (Stone, Naimark, Ambrose, Godement) theorem.

Theorem 5.2 (S.N.A.G.)
Any representation of a locally compact abelian group G is a direct sum of representations U_μ where μ is a Borel measure on $\widehat{G}$.

Examples 5.3
1) Let $U : R \to Aut(H)$ be a representation of the real line. Then $U = \bigoplus_{i=1}^{\infty} U_{\mu_i}$, and

$$(U_{\mu_i}(g)\phi)(\hat{g}) = \hat{g}(g)\phi(\hat{g})$$

where the Hilbert space is $L^2(\widehat{R}, \mu_i)$. But the isomorphism between $\widehat{R}$ and R given by $p \to e^{ipt}$ allows us to replace $L^2(\widehat{R}, \mu_i)$ with a more familiar space. Simply use the change of measure provided by the map $T : p \to e^{ipt}$ and $\sigma_i(E) = \mu_i(T(E))$. Then the representation U_{μ_i} is equivalent to the representation $U_i : R \to L^2(R, \sigma_i)$ defined by

$$(U_i(p)\phi)(x) = e^{ipx}\phi(x).$$

The space becomes $L^2(R, \sigma_i)$.

We can express this representation in terms of the spectral theorem in the following way. An element in the direct sum is a sequence $\{f_i\}$ of functions defined on R. We define a projection valued measure on R by

$$P(E)(f_1, f_2, \ldots) = (\chi_E f_1, \chi_E f_2, \ldots),$$

where χ_E is the characteristic function of the set E. Then

$$\langle U(p)\{f_i\}, \{h_i\}\rangle = \sum_{i=1}^{\infty} \int_R e^{ipx} f_i(x) \bar{h_i}(x)\, d\sigma_i.$$

But

$$\langle P(E)f_i, h_i\rangle = \int_E f_i(x)\bar{h}_i(x)\, d\sigma_i,$$

and so

$$\langle U(p)\{f_i\}, \{h_i\}\rangle = \int_R e^{ipx}\, d\langle P()\{f_i\}, \{h_i\}\rangle.$$

2) This may be repeated almost word for word for R^4. The points in R^4, (E, p_1, p_2, p_3), are interpreted as points in enery–momentum space. In this case, the measures σ_i are often supported on a mass shell $p_1^2 + p_2^2 + p_3^2 = E^2$. •

6 Representations of Semi–Direct Products

In this section we describe all the representations of semi–direct products of the form $H \bar{\times}_\tau A$ where A is abelian. This includes the Poincar e group. The strategy is reasonable enough. We shall relate the representations of $H \bar{\times}_\tau A$ to the representations of the factors, and then we shall show how H always induces an action on $\widehat{A}$. Thus the representations of H can be induced from representations of little groups.

Before begining, recall that the operation in a semi–direct product is defined as follows. We have the homomorphism $\tau : H \to Aut(A)$, then

$$(h_1, a_1)(h_2, a_2) = (h_1 h_2, a_1 \tau_{h_1}(a_2)).$$

Let $G = H \bar{\times} A$ (τ is understood), and $W : G \to Aut(K)$ a representation of G. Let $V : H \to Aut(K)$ and $U : A \to Aut(K)$ be the restrictions of W to H and A respectively. Let $\widehat{A}$ be the dual of A.

Observe that H acts on $\widehat{A}$ through $h\hat{g}(a) = \hat{g}(\tau_h(a)$. We shall see shortly why we chose the action on $\widehat{A}$ rather than the more obvious action on A.

It will be useful to note that with H identified with $H \times \{e\}$ and A identified with $\{e\} \times A$ that

$$\begin{aligned} hah^{-1} &= (h, e)(e, a)(h, e)^{-1} \\ &= (h, \tau_h a)(h^{-1}, e) \\ &= (e, \tau_h(a)\tau_h(e)) = \tau_h(a). \end{aligned}$$

So, with these identifications , τ_h is an inner automorphism of A.

Now consider the restrictions U and V of W. We have

$$V_h U_a V_{h^{-1}} = W_{hah^{-1}} = U_{hah^{-1}}.$$

But the S.N.A.G. theorem provides a unique projection valued measure P on $\widehat{A}$ such that

$$U_a = \int_{\widehat{A}} \hat{g}\, dP(\hat{g}).$$

Now $U_{hah^{-1}}$ also defines a representation of A, but then the change of measure induced by $\hat{g} \to h\hat{g}$ and $\bar{P}(E) = P(hE)$ we have

$$\begin{aligned}
\langle U_{hah^{-1}}\phi, \psi\rangle &= \int_{\widehat{A}} \hat{g}(hah^{-1})\, d\langle P()\phi, \psi\rangle \\
&= \int_{\widehat{A}} \hat{g}(a)\, d\langle P(h^{-1}()h)\phi, \psi\rangle \\
&= \int_{\widehat{A}} \hat{g}(a)\, d\langle \bar{P}()\phi, \psi\rangle.
\end{aligned}$$

But

$$\begin{aligned}
\langle U_{hah^{-1}}\phi, \psi\rangle &= \langle U_a V_h^{-1}\phi, V_h^{-1}\psi\rangle \\
&= \int_{\widehat{A}} \hat{g}(a)\, d\langle V_h P() V_h^{-1}\phi, \psi\rangle.
\end{aligned}$$

Since the measure is unique we have

$$V_h P(E) V_h^{-1} = P(hE),$$

and so (V, P) is a system of imprimitivity on $\widehat{A}$.

Conversely, let U and V be representations of A and H in a Hilbert space K. Let P be the projection valued measure on $\widehat{A}$ provided by the S.N.A.G. theorem, and assume that (V, P) is a system of imprimitivity on $\widehat{A}$. Define

$$W_{(h,a)} = U_a V_h = W_{ah},$$

where $ah = (e, a)(h, e) = (h, a)$. Then we have

$$\begin{aligned}
W_{a_1 h_1 a_2 h_2} &= W_{a_1 (h_1 a_2 h_1^{-1}) h_1 h_2} \\
&= U_{a_1 (h_1 a_2 h_1^{-1})} V_{h_1 h_2} \\
&= U_{a_1} V_{h_1} U_{a_2} V_{h_1}^{-1} V_{h_1} V_{h_2} \\
&= W_{a_1 h_1} W_{a_2 h_2}.
\end{aligned}$$

It is not immediately obvious that $U_{a_1(h_1 a_2 h_1^{-1})} = V_{h_1} U_{a_2} V_{h_1}^{-1}$, but

$$\begin{aligned} \langle U_{hah^{-1}}\phi, \psi\rangle &= \int_{\widehat{A}} \hat{g}(hah^{-1})\, d\langle P()\phi, \psi\rangle \\ &= \int_{\widehat{A}} \hat{g}(a)\, d\langle V_h P() V_h^{-1}\phi, \psi\rangle \\ &= \langle V_h U_a V_h^{-1}\phi, \psi\rangle \end{aligned}$$

by change of measure and the fact that (V, P) is a system of imprimitivity. We now have the following theorem.

Theorem 6.1
Let U and V be representations of A and H in a Hilbert space K respectively. Let P be the projection valued measure corresponding to U given by the S.N.A.G. theorem. Then a necessary and sufficient condition that there exists a representation W of $H \hat{\times} A$ whose restrictions are U and V is that (V, P) be a system of imprimitivity based on $\widehat{A}$.

Theorem 6.2
A representation W of $H \hat{\times} A$ is irreducible if and only if the corresponding system of imprimitivity (V, P) is irreducible. Moreover, two representations W_1 and W_2 are equivalent if and only if the corresponding systems of imprimitivity are equivalent.

Proof: Suppose W is reducible, then there is a projection Q that commutes with V and with U. Thus Q commutes with P, and so (V, P) is reducible. The converse follows in the same way, since then Q commuting with P implies Q commutes with U, and thus with W. •

We are not yet quite in a position to describe all of the representations of $H \bar{\times} A$ because the action of H on $\widehat{A}$ is not necessarily transitive, so the action breaks up $\widehat{A}$ into orbits. To see how the orbit structure plays a role, we give an example.

Example 6.3
Let $G = S^1 \bar{\times} R^2$ be the Euclidean group acting on R^2. Observe that

$$\widehat{R^2} = \{\hat{g} | \hat{g}(x, y) = e^{i(p_1 x + p_2 y)} \text{ for some } (p_1, p_2) \in R^2\}.$$

If we identify each $\hat{g}$ with the corresponding (p_1, p_2), then we see that the action of S^1 is rotation in the plane. This is not transitive, but each orbit is a circle or the origin. Consider the orbit of radius $p > 0$ (we consider $p = 0$ shortly). The only invariant measures on this orbit are multiples of the arclength measure. Let μ be the arclength measure. The stabilizer (little group) of any point in the orbit is the identity subgroup $\{e\}$. The inducing construction now gives us all the irreducible systems of imprimitivity by inducing from the irreducible representations of $\{e\}$! These are represented by the complex numbers of length one. The representation space is $L^2(C_p, \mu)$ where C_p is the orbit of radius p, and so the induced representation is defined by

$$(V_r\phi)(x) = L(c(x)^{-1}rc(r^{-1}x))\phi(r^{-1}x)$$

where $c(x)$ is the rotation that rotates the fixed point x_0 to $x \in C_p$. Thus $c(x)^{-1}rc(r^{-1}x) = e$, and so L only multiplies by a fixed constant of length one, so we may set it to one. Thus

$$(V_r\phi)(x) = \phi(r^{-1}x).$$

The corresponding spectral projection is

$$(P(S)\phi)(x) = \chi_S\phi(x),$$

and by Mackey's theorem the system of imprimitivity is irreducible.

The projection valued measure P can be expanded to the plane by $P(E) = P(E \cap C_p)$, and likwise μ can be extended to the plane by

$$\mu = \delta(p - \sqrt{p_1^2 + p_2^2}).$$

The pair (V, P) remains an irreducible system of imprimitivity. We define U on R^2 by

$$\begin{aligned} \langle U_r\phi, \psi\rangle &= \int_{R^2} e^{ip\cdot r}\, d\langle P()\phi, \psi\rangle \\ &= \int_{R^2} e^{ip\cdot r}\phi\bar{\psi}\delta(p - \sqrt{p_1^2 + p_2^2})\, dp_1 dp_2. \end{aligned}$$

Thus

$$(W_{ah}\phi)(\bar{p}) = U_a V_h \phi(\bar{p}) = e^{ip\cdot a}\phi(h^{-1}\bar{p}).$$

We now consider the case where $p = 0$. In this case the origin is the fixed point, and the little group is S^1. The irreducible representations of S^1 are the group's characters. The measure is concentrated on $\{0\}$, and up to equivalence we may take the measure to have value one. Thus $U_a = I$, and

$$W_{ah} = IV_h = V_h.$$

But V is a character of S^1, so

$$V_z = z^n$$

where $|z| = 1$ and n is an integer. •

This example has a special feature which ensures that the method described in the example generates all solutions of the semi–direct product. All the orbits are cut uniquely by a ray from the origin.

Definition 6.4
We say that a G–space X has a smooth orbit structure if there is a Borel subset of X that intersects each orbit exactly once. •

Before stating the theorem note that if X is an H–space and w is an orbit and E is a subset of X then it is easy to prove that

$$(hE) \cap w = h(E \cap w).$$

Therefore, if (V, P) is a system of imprimitivity based on w, then P can be extended to the Borel subsets of X by $P(E) = P(E \cap w)$, and (V, P) remains a system of imprimitivity.

Theorem 6.5 (Mackey)
Let $G = H \bar{\times} A$. Let w be an H–orbit in $\widehat{A}$. Let x_w be a point in w and m an irreducible representation of the stability subgroup of x_w. Let V be the induced representation. There is a projection valued measure P on the orbit w such that (V, P) is a system of imprimitivity, and this system extends to $\widehat{A}$. Define

$$U_a = \int_{\widehat{A}} \hat{g}(a)\, dP.$$

Then $W^{m,w}_{ah} = U_a V_h$ is irreducible. Moreover, W^{m_1,w_1} is equivalent to W^{m_2,w_2} if and only if $w_1 = w_2$ and m_1 is equivalent to m_2. If the H–orbit structure of $\widehat{A}$ is smooth, then each irreducible representation of G is equivalent to some $W^{m,w}$.

7 Representations of the Poincaré Group

We are now in a position to discuss the representations of the Poincaré group $G = L_+^\uparrow \hat{\times} R^4$. We can identify $\widehat{R^4}$ with R^4, and then the action of $L_+^\uparrow$ on $p = (p_0, p_1, p_2, p_3)$ is Λp. We consider the representations that have mass shells for orbits and then those that use the forward light cone as an orbit.

7.1 The Mass Shell and Localization

Consider the mass shell

$$p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2,$$

and the stability group of $(m, 0, 0, 0)$ – $SO(3)$.

We first need a quasi–invariant or invariant measure on the mass shell M. The mass shell is a hypersurface in R^4, so we can take a tip from surfaces in R^3. Surface area for a surface S is provided by the "volume form" $\omega = \sqrt{det(g_{ij})}dx^1 \wedge dx^2$ where (g_{ij}) is the matrix of the ordinary inner product restricted to the tangent spaces of S. Now if the surface S is an orbit under the action of $SO(3)$, the action preserving the dot product, then areas of regions on this surface moved by the action will be preserved. In our case, the action is that of $L_+^\uparrow$, and so the "inner product" will be the Lorentz metric instead of the Riemannian metric. If we choose the space coordinates for our coordinates on the manifold M, then the matrix (g_{ij}) is the matrix of the Lorentz metric relative to these coordinates. After a little calculation we get the form

$$\omega = \frac{m}{\sqrt{\bar{p}^2 + m^2}} dp_1 \wedge dp_2 \wedge dp_3$$

where $\bar{p} = (p_1, p_2, p_3)$. If we pull this down to the coordinate space we get the following definition of our invariant measure.

Let E be a Borel subset of M, and let $\bar{E}$ be the projection of E onto the coordinates (p_1, p_2, p_3). Define

$$\mu(E) = \int_{\bar{E}} \frac{1}{\sqrt{\bar{p}^2 + m^2}}\, d^3\bar{p}.$$

Note that the numerator has been set to 1 without loss of generality. We now show directly that this measure is invariant. Clearly, it is invariant under $SO(3)$. Let Λ be the boost defined by

$$\begin{aligned} p_1' &= \gamma(p_1 - vp_0) \\ p_2' &= p_2 \\ p_3' &= p_3 \\ p_0' &= \gamma(p_0 - vp_1) \end{aligned}$$

where $\gamma = 1/\sqrt{1-v^2}$. Let $E' = \Lambda E$, then

$$\mu(E') = \int_{\bar{E}'} \frac{1}{2\sqrt{\bar{p}^2 + m^2}}\, d^3\bar{p}.$$

If $\bar{p} \in \bar{E}$ then $(\sqrt{\bar{p}^2+m^2}, \bar{p}) \in E$. Thus

$$\begin{aligned} p_1' &= \gamma(p_1 - v\sqrt{\bar{p}^2 + m^2}) \\ p_2' &= p_2 \\ p_3' &= p_3 \end{aligned}$$

defines the transform $\bar{\Lambda}$ of $\bar{E}$ to $\bar{E}'$. The Jacobian of this transformation is

$$\gamma(1 - \frac{vp_1}{\sqrt{\bar{p}^2 + m^2}}) = \gamma(1 - \frac{vp_1}{p_0}).$$

Thus by change of variables

$$\mu(E') = \int_{\bar{E}'} \frac{1}{p_0}\, d^3\bar{p} = \int_{\bar{E}} \frac{1}{\sqrt{(\bar{\Lambda}\bar{p})^2 + m^2}} |\gamma(1 - \frac{vp_1}{p_0})|\, d^3\bar{p}$$

$$= \int_{\bar{E}} \frac{1}{\gamma(p_0 - vp_1)} \gamma \frac{p_0 - vp_1}{p_0}\, d^3\bar{p} = \mu(E).$$

Now let $L : SO(3) \to Aut(K)$ be an irreducible representation of the stability subgroup of $x_0 = (m, 0, 0, 0)$. This is a finite dimensional representation, and we will discuss these representations when we consider projective representations. We must choose a Borel section for the inducing consruction. For the point $(p_0, \bar{p})$ on the mass shell, we let $c((p_0, \bar{p})) = R(\bar{p})B(|\bar{p}|)R^{-1}(\bar{p})$ where $R(\bar{p})$is a standard rotation that takes the 1–axis into the direction $\bar{p}$ and $B(|\bar{p}|)$ is the boost along the 1–axis direction that takes $(m,0,0,0)$to $(p_0,\bar{p})$. We now have the induced representation

$$(V_\Lambda\phi)(p_0,\bar{p}) = L(c(p_0,\bar{p})^{-1}\Lambda c(\Lambda^{-1}(p_0,\bar{p}))\phi(\Lambda^{-1}(p_0,\bar{p})).$$

Here, $\phi\in L^2(M,K,\mu)$. The projection on the mass shell is defined by

$$(P(E)\phi)(p_0,\bar{p}) = \chi_E(p_0,\bar{p})\phi(p_0,\bar{p}).$$

We can extend both P and μ to all of R^4 by setting $P(E) = P(E\cap M)$ and $\mu(E) = \mu(E\cap M)$. With these extensions, the pair (V,P) is a system of imprimitivity based on R^4. Finally, we define U by

$$\langle U_a\phi,\psi\rangle = \int_{R^4}\hat{g}(a)\,d\langle P()\phi,\psi\rangle,$$

where

$$\langle P(E)\phi,\psi\rangle = \int_E\langle\phi,\psi\rangle_K\,d\mu.$$

Thus

$$U_a\phi(p_0,\bar{p}) = e^{i(\sqrt{\bar{p}^2+m^2}a_0+\bar{p}\cdot\bar{a})}\phi(p_0,\bar{p}).$$

Therefore, the representation of $L_+^\uparrow\bar{\times}R^4$ is

$$W_{(\Lambda,a)}\phi(p_0,\bar{p}) = U_aV_\Lambda\phi(p_0,\bar{p})$$

$$= e^{i(\sqrt{\bar{p}^2+m^2}a_0+\bar{p}\cdot\bar{a})}L(c(p_0,\bar{p})^{-1}\Lambda c(\Lambda^{-1}(p_0,\bar{p}))\phi(\Lambda^{-1}(p_0,\bar{p})).$$

We now seek position observables in this representation. To do this we will replace the space $L^2(R^4,K,\mu)$ with unitarily equivalent spaces.

The space $L^2(R^4,K,\mu)$ is unitarily equivalent to $L^2(R^3,K,\frac{d^3\bar{p}}{\sqrt{\bar{p}^2+m^2}})$. The only difference in the representation $W_{(\Lambda,a)}$ is that $\phi(p_0,\bar{p}) = \psi(\bar{p})$ and $\Lambda^{-1}\bar{p}$ is defined to be the projection of $\Lambda^{-1}(p_0,\bar{p})$ into R^3. The restriction of this representation to the Euclidean subgroup is

$$W_{(R,\bar{a})} = e^{i\bar{p}\cdot\bar{a}}\phi(R^{-1}\bar{p}).$$

A second equivalence is between $L^2(R^3,K,\frac{d^3\bar{p}}{\sqrt{\bar{p}^2+m^2}})$ and $L^2(R^3,K,d^3\bar{p})$ given by

$$(T\phi)(\bar{p}) = \frac{1}{(\bar{p}^2+m^2)^{\frac{1}{4}}}\phi(\bar{p}).$$

Define the equivalent representation by $\bar{W}_{(R_0,\bar{a})} = TW_{(R_0,\bar{a})}T^{-1}$, then

$$(\bar{W}_{(R_0,\bar{a})}\phi)(\bar{p}) = e^{i\bar{p}\cdot\bar{a}}L(c(\bar{p})^{-1}R_0c(R_0^{-1}\bar{p})\phi(R_0^{-1}\bar{p}).$$

We now want to use the Fourier transform as a third equivalence, but for this to succeed we shall see that $c(\bar{p})^{-1}R_0c(R_0^{-1}(\bar{p})$ is independent of $\bar{p}$, and is in fact R_0 where this is a pure rotation. We have

$$c(\bar{p})^{-1}R_0c(R_0^{-1}(\bar{p}) = R(\bar{p})B^{-1}R^{-1}(\bar{p})R_0R(R_0^{-1}\bar{p})BR^{-1}(R_0^{-1}\bar{p}).$$

But

$$R^{-1}(\bar{p})R_0R(R_0^{-1}\bar{p}) = R(\theta)$$

where $R(\theta)$ is a pure rotation about the 1–axis. This commutes with B, so

$$c(\bar{p})^{-1}R_0c(R_0^{-1}(\bar{p}) = R(\bar{p})R(\theta)R^{-1}(R_0^{-1}\bar{p}) = R_0.$$

The representation of the Euclidean group is now

$$(\bar{W}_{(R_0,\bar{a})}\phi)(\bar{p}) = e^{i\bar{p}\cdot\bar{a}}L(R_0)\phi(R_0^{-1}\bar{p}).$$

The quantity $L(R_0)$ is a matrix which is constant with respect to $\bar{p}$, and so it commutes with the Fourier transform. We define

$$\hat{W}_{(R_0,\bar{a})} = \mathcal{F}^{-1}\bar{W}_{(R_0,\bar{a})}\mathcal{F}$$

where $\mathcal{F}$ is the Fourier transform. Now we have

$$(\hat{W}_{(R_0,\bar{a})}\phi)(\bar{x}) = \mathcal{F}^{-1}(e^{i\bar{p}\cdot\bar{a}}L(R_0)\hat{\phi}(R_0^{-1}\bar{p})$$

$$= \int_{R^3} e^{i(\bar{p}\cdot\bar{a}+\bar{p}\cdot\bar{x})}L(R_0)\hat{\phi}(R_0^{-1}\bar{p})\,d^3\bar{p}.$$

On the other hand,

$$(\mathcal{F}^{-1}L(R_0)\hat{\phi}(R_0^{-1}\bar{p}))(\bar{x}+\bar{a}) = \int_{R^3} e^{i\bar{p}\cdot(\bar{x}+\bar{a})}L(R_0)\hat{\phi}(R_0^{-1}\bar{p})\,d^3\bar{p},$$

so

$$\hat{W}_{(R_0,\bar{a})}\phi)(\bar{x}) = L(R_0)\phi(R_0^{-1}(\bar{x}+\bar{a}).$$

Therefore, we can interpret $||\phi||^2$ as the probability density of position of the particle since this density transforms correctly under the Euclidean group.

7.2 The Forward Light Cone

We now consider representations that have the forward light cone as the orbit of $L_+^\uparrow$ and $S^1 \bar{\times} R^2$ as the little group of, say, $x_0 = (1,0,0,1)$. There are interesting differences between this case and the preceding one.

We have seen that the representations of the little group are of two types depending on if the orbit is the origin or is a circle. In the first case the irreducible representations have the form $L((z,a)) = V_z = z^n$ where n is an integer, a character of S^1. Here the rotation is represented by multiplication by a complex number z of length 1. The integer n is called the helicity.

The next step in the inducing construction is to define a Borel section c. We cannot use $c((p_0,\bar{p})) = R(\bar{p})B(|\bar{p}|)R^{-1}(\bar{p})$, the one we used in the massive case, since this will not send $(1,0,0,1)$ to $(p_0,\bar{p})$. However, $c((p_0,\bar{p})) = R(\bar{p})B(|\bar{p}|)$, which first boosts along the 1–axis and then rotates to the $\bar{p}$ direction, does send $(1,0,0,1)$ to $(p_0,\bar{p})$. But, and this is a crucial difference, for a pure rotation R_0

$$c(\bar{p})^{-1}R_0c(R_0^{-1}(\bar{p}) = B^{-1}R^{-1}(\bar{p})R_0R(R_0^{-1}\bar{p})B = R(\theta)$$

where $R(\theta)$ is a rotation about the 1–axis and is dependent on $\bar{p}$. Thus $L(c(\bar{p})^{-1}R_0c(R_0^{-1}(\bar{p}))$ is dependent on $\bar{p}$ except in the case where the helicity n equals zero. In this case, the derivation of a position representation procedes exactly as before. However, the Fourier transform step will fail for non–zero helicities, so one does not expect position representations for these particles to exist.

Consider the zero helicity case again. In the position representation, the observable of position in the i^{th} direction, X_i, is multiplication by x_i. Thus, under the inverse Fourier transform in the momentum representation, the position operator is $\frac{1}{i}\frac{\partial}{\partial p_i}$. The Hamiltonian in this representation is multiplication by $H = \sqrt{\bar{p}^2}$. The i^{th} component of velocity of the mean at $t = 0$ is the derivative of

$$\langle e^{iHt}\frac{1}{i}\frac{\partial}{\partial p_i}e^{-iHt}\phi, \phi\rangle$$

at $t = 0$, and this is

$$\langle i(H\frac{1}{i}\frac{\partial}{\partial p_i} - \frac{1}{i}\frac{\partial}{\partial p_i}H)\phi, \phi\rangle.$$

This simplifies to

$$\frac{p_i}{\sqrt{\bar{p}^2}},$$

and so the speed, which is the square root of the sum of the squares of these terms, is the identity. Therefore, the speed of the mean position (group velocity?)of the particle is always 1, which in our case is the speed of light.

The second type of representation of $ISO(2)$ is infinite dimensional, and there do not seem to be any corresponding particles of nature. Therefore, we will drop this case.

8 Multiplier Representations and Covering Groups

We have been dealing with pure representations up to now, but the physically most general representations of symmetry groups are the projective or multiplier representations. Such representations have the relation

$$U_g U_h = \omega(g,h) U_{gh}$$

where $|\omega(g,h)| = 1$. This is because $U_g U_h$ and U_{gh} implement the same symmetry of the ray space of the Hilbert space in question and so they can only differ by a multiple of modulus 1.

The first step is to relate the multiplier representations of a group to the true representations of its simply connected covering group. Let $R : G \to H$ be the covering of H by the simply connected group G. Now let $U : H \to Aut(K)$ be an irreducible representation of H, then $\widehat{U} = U \circ R$ is an irreducible representation of G.

Conversely, let $\widehat{U} : G \to Aut(K)$ be an irreducible representation of G. Assume that $Ker(R)$ is in the center of G (for all cases we consider, it is). Then for $g \in Ker(R)$, $\widehat{U}_g$ is in the intertwining ring of $\widehat{U}$, and so

$$\widehat{U}_g = \chi_g I$$

by Schur's lemma. Let $h \in H$ and $R(g) = R(g') = h$. Define

$$T_h(M) = \widehat{U}_g(M) = \{\widehat{U}_g(\phi) \mid \phi \in M\},$$

where M is a closed subspace of K. But $\widehat{U}_{gg'^{-1}} = \chi_{gg'^{-1}} I$, so $\widehat{U}_g = \chi_{gg'^{-1}} \widehat{U}_{g'}$, so $\widehat{U}_{g'}(M) = \widehat{U}_g(M)$. Therefore, T_h is a well–defined projective representation of H. Let U_h be the corresponding unitary implementation. Then U is a multiplier representation with multiplier ω. So for each irreducible representation of G we have a corresponding multiplier representation of H

$$\widehat{U} \to U.$$

On the other hand, let U be an irreducible ω–representation of H. Then $\widehat{U} = U \circ R$ is an ω^*–representation of G where $\omega^*(g,h) = \omega(Rg, Rh)$. But G is simply connected, so $\widehat{U}$ is equivalent to a true representation $\widehat{V}$. Let V be the multipier representation of H corresponding to $\widehat{V}$. Then $V_h(M) = \widehat{V}_g(M)$ where $R(g) = h$. But $\widehat{V}_g(M) = \widehat{U}_g(M) = U_h(M)$, and so U and V implement the same projective representation and therefore have equivalent multipliers.

In short, the irreducible multiplier representations of H are those derived from the irreducible true representations of the simply connected covering group G in the manner just described.

We use the above correspondence to discuss the multiplier representations of the Poincaré group. We have noted that

$$\Lambda : SL(2,C) \to L_+^\uparrow$$

defined by

$$\Lambda(A)(x) = \tau^{-1}(A\tau(x)A^*)$$

is a double covering of $L_+^\uparrow$. It is not hard to show that

$$Ker(\Lambda) = \{\pm I\}.$$

The map

$$R = \Lambda \times I : SL(2,C) \bar{\times} R^4 \to L_+^\uparrow \bar{\times} R^4$$

is a double covering of the semi–direct product $L_+^\uparrow \bar{\times} R^4$ by $SL(2,C)\bar{\times} R^4$. In the latter, the action of $SL(2,C)$ on R^4 is by $Aa = \Lambda(A)a$. The kernel of this map is

$$Ker(\Lambda \times I) = \{(\pm I, 0)\},$$

and so it is in the center as is required by the analysis above. Thus the projective representations of $L_+^\uparrow \bar{\times} R^4$ can be derived from the true representations of $SL(2,C)\bar{\times} R^4$.

Let U be a multiplier representation on $L_+^\uparrow \bar{\times} R^4$ with multiplier ω. Let $\widehat{U}$ be the corresponding true representation as described above. We distinguish two cases. We have $\widehat{U}_{(-I,0)}\widehat{U}_{(-I,0)} = \widehat{U}_{(I,0)} = I$, so $\widehat{U}_{(-I,0)}$ is one of the two square roots of I, namely $\pm I$. We consider these two posibilities in turn.

We choose a section $S : L_+^\uparrow \bar{\times} R^4 \to SL(2,C)\bar{\times} R^4$, i.e. $R(S(h)) = h$. Observe that

$$R(S(hh')) = hh' = R(S(h)S(h')),$$

and so

$$S(hh') = \pm S(h)S(h').$$

If $\widehat{U}_{(-I,0)} = I$ we define a true representation on $L_+^\uparrow \bar{\times} R^4$ by

$$V_h = \widehat{U}_{S(h)}.$$

We have

$$V_{hh'} = \widehat{U}_{S(hh')} = \widehat{U}_{\pm(I,0)S(h)S(h')} = \widehat{U}_{\pm(I,0)}\widehat{U}_{S(h)}\widehat{U}_{S(h')} = V_h V_{h'},$$

thus the representation is a true one.

If $\widehat{U}_{(-I,0)} = -I$ then we define

$$\bar{\omega}(h,h') = \begin{cases} 1 & \text{if } S(hh') = S(h)S(h') \\ -1 & \text{if } S(hh') = -S(h)S(h'). \end{cases}$$

If $S(hh') = S(h)S(h')$ we have

$$\bar{\omega}(h,h')V_{hh'} = V_{hh'} = \widehat{U}_{S(hh')} = \widehat{U}_{S(h)S(h')} = V_h V_{h'},$$

but if $S(hh') = -S(h)S(h')$, then

$$\bar{\omega}(h,h')V_{hh'} = -V_{hh'} = -\widehat{U}_{S(hh')} = -\widehat{U}_{-(I,0)S(h)S(h')} = V_h V_{h'}.$$

Therefore, we have a multiplier representation with multiplier $\bar{\omega}$. This multiplier is cohomologous with ω since they belong to multiplier representations that implement the same projective representation.

It is worth noting that $S(hh') = -S(h)S(h')$ for some h and h', for if $S(hh') = S(h)S(h')$ for all h and h', then S splits the short exact sequence

$$0 \to \{I, -I\} \to SL(2,C)\bar{\times} R^4 \to L_+^\uparrow \bar{\times} R^4 \to 0.$$

But then $SL(2,C)\bar{\times} R^4$ is isomorphic and homeomorphic to $L_+^\uparrow \bar{\times} R^4 \times \{I, -I\}$. But, the latter is disconnected whereas the former is connected. This is impossible.

We can now find all the multiplier representations of the Poincaré group by finding all the true representations of $SL(2,C)\bar{\times}R^4$. But, these latter representations are found by the inducing construction exactly as before. For the mass shell the little group becomes $SU(2)$.

Example 8.1 (Integer and Half–Integer Spin)
The irreducible representations of $SU(2)$ are all finite dimensional and easy to describe. Consider the vector space of a complex homogeneous polynomials of degree n

$$P(z_1, z_2) = \sum_{k=0} a_k z_1^{n-k} z_2^k.$$

This space has dimension $n+1$. The action of $SU(2)$ is defined as follows. Let

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and define $L(g)P = gP$ where

$$gP(z_1, z_2) = P((z_1, z_2)\begin{bmatrix} a & b \\ c & d \end{bmatrix}).$$

One can define an inner product so that this action is unitary.

The action of $SL(2,C)$ on the point $(m,0,0,0)$ produces the mass shell as before. The induced representation is

$$(W(g,a)\phi)(\bar{p}) = e^{i(\sqrt{\bar{p}^2+m^2}a_0+\bar{p}\cdot\bar{a})}L(c(\bar{p})^{-1}gc(g^{-1}\bar{p}))\phi(g^{-1}\bar{p}).$$

Now

$$(W_{(-I,0)}\phi)(\bar{p}) = L(-I)\phi(\bar{p}).$$

But,

$$L(-I)P(z_1, z_2) = (-1)^n P(z_1, z_2),$$

and so $L(-I) = \pm I$ when n is even or odd. Therefore, if n is even then $W_{(-I,0)} = I$ and the corresponding representation of $L_+^\uparrow \bar{\times} R^4$ is a true representation. If n is odd then $W_{(-I,0)} = -I$, and the representation of $L_+^\uparrow \bar{\times} R^4$ is a multiplier representation as described above.

The true representations are refered to as the integer spin representations and the multiplier representations are termed the half–integer spin representations. •