Additional Exercises

These additional exercises are a work in progress, and they will be added to as time permits.

Chapter 1

1. Show that $y(x) = 3\cos(x)$ is a solution to $y'' + y = 0$. What is the interval of solution?

2. Show that $y = e^{x^2/2}$ solves $y' = xy$. What is the interval of solution?

3. Show that $y = 1 + e^{3x}$ is a solution to $y'' - 2y' - 3y = -3$.

4. Show that $y = \frac{1}{x-1}$ is a solution to $y' = -y^2$ on $(1, \infty)$.

Translate the statements in Exercises 5–7 into differential equations.

5. The acceleration of a particle is equal to a constant g minus the velocity of the particle.

6. The rate of change of atmospheric pressure with respect to height is equal to the negative of the pressure.

7. The rate of change of a population of bacteria is proportional to the cube of the population minus the square of the population.

8. Show that the functions $u_1 = e^x + 2xe^x$ and $u_2 = e^x + 2e^x + 2xe^x$ satisfy the system

$$\begin{aligned} u_1'(x) &= u_2(x) \\ u_2'(x) &= -u_1(x) + 2u_2(x). \end{aligned}$$

Reduce the equations in 9–11 to a first order system.

9. $y^{(3)} - 2y'' + y = 0.$

10. $y'' + 3xy' + 4(y)^2 = e^x.$

11. $y^{(4)} - 3y'' - y = 0.$

12. Suppose $\phi(x)$ satisfies $y' = -y^2 - 1$. What can you say qualitatively about the behavior of $\phi(x)$. Hint: the sign of the derivative is the key.

13. It is easy to check that $\phi(x) = e^{2x}$ satisfies $y' = 2y$. Would you describe this solution as analytical, graphical, or numerical? Explain!

14. Use the software package you have chosen to graph (a) $f(x) = \sin(x^2)$, (b) $f(x) = \frac{x^2+1}{x-1}$, and (c)$f(x) = \arcsin(4x)$.

15. Use the software package you have chosen to graph (a) $f(x) = \tan(4x)$, (b)$f(x) = x\sin(3x)$, and (c) $f(x) = 3x^2 - 4x$.

Section 2.1

1. Draw the field of $y' = 5(y^2)^{1/3}$ in the window $[-6, 6] \times [-4, 4]$. Draw solutions through $(0, 0)$ and through $(-2, -1)$. What does Theorem 2.1.2 have to say about this? Explain!

2. Draw the field of $y' = 2((y - 1)^2)^{1/3}$ in the window $[-6, 6] \times [-4, 4]$. Draw solutions through $(0, 1)$ and through $(0, 0)$. What does Theorem 2.1.2 have to say about this? Explain!

3. Use Field&Solution to draw solutions through $(0,0)$ and $(0,-0.1)$ for the equation $y' = f(x,y) = |y|^{2/5}$. Without any calculation, what can you say about $\partial f/\partial y$ and why?

4. Use Field&Solution to draw solutions through $(0,0)$ and $(0,1)$ for the equation $y' = f(x,y) = |y-1|^{1/3}$. Without any calculation, what can you say about $\partial f/\partial y$ and why?

5. Without using the Field&Solution package, describe the behavior of solutions to $y' = (y-1)^4$ in general terms. Is there a constant solution? Hint: What does the derivative tell you?

6. Without using the Field&Solution package, describe the behavior of solutions to $y' = -y^2$ in general terms. Is there a constant solution? Hint: What does the derivative tell you?

Section 2.2

1. By hand, carry out one step of Euler's method and the Runge–Kutta method where $y' = 2xy - y$, $y(1) = 2$, and the step size is 0.1.

2. By hand, carry out one step of Euler's method and the Runge–Kutta method where $y' = x^2y - y$, $y(0) = 2$, and the step size is 0.2.

In Exercises 3–4, use the package ERGraphical.

3. Consider the equation $y' = x - 2y$. Approximate the solution on $[-2,3]$ with initial value $y_0 = 0$ using the Euler and Runge–Kutta methods with step sizes $h = 1$, $h = 0.5$, and $h = 0.1$. Compare the performance of the two methods.

4. Consider the equation $y' = \frac{y^3}{2}$. Approximate the solution on $[0,1]$ with initial value $y_0 = 1$ using the Euler and Runge–Kutta methods with step sizes $h = 1$, $h = 0.5$, $h = 0.1$, and $h = 0.01$. Use a window of about $[-1,1] \times [-15,15]$ Compare the performance of the two methods.

In each of the exercises 5–6 use the package ERNumerical.

5. Consider the equation $y' = x + 2y$. Approximate the solution on $[0, 2]$ with initial value $y_0 = 2$. Run the package for step sizes of $h = 0.1$, $h = 0.05$, $h = 0.02$, and $h = 0.01$. Record the values of both methods at $x = 1$. What approximate value are you confident in for each method and why?

6. Consider the equation $y' = (x^2 + 2y)/4$. Approximate the solution on $[0, 2]$ with initial value $y_0 = 2$. Run the package for step sizes of $h = 0.1$, $h = 0.05$, $h = 0.02$, and $h = 0.01$. Record the values of both methods at $x = 1$. What approximate value are you confident in for each method and why?

Section 2.3.4

In Exercises 1–4 check to see if the equation is exact in the plane. If so, solve it and clearly indicate on a plot of the contour the extent of the solution through the initial condition.

1. $(2x + 3) + (2y - 2)y' = 0$ and $y(0) = 1$.

2. $(2x^2 - 2xy + 4) + (5y^2 - x^2 + 5)y' = 0$ and $y(1) = -1$.

3. $(x^2y + y) + (x^3/3 + y)y' = 0$ and $y(0) = 1$.

4. $(9x^2 + y - 1) + (x - 4y)y' = 0$ and $y(1) = 3$.

Orthogonal trajectories. Each equation $y' = f(x, y)$ determines a family of curves in the plane. Exercises 5–8 ask you to find a second family of curves such that each curve of this family intersects the curves of the first family at right angles. Such a family is called the **family of orthogonal trajectories**.

5. Find the orthogonal trajectories of the family $y = cx^2$. Hint: find a first order differential equation of which the curves of this family are solutions. Do this by writing $y' = 2cx$, and now substitute for c from the family.

6. Find the orthogonal trajectories of the family $x^2 + y^2 = c$. Hint: see 5.

7. Find the orthogonal trajectories of the family $y = ce^x$. Hint: see 5.

8. Find the orthogonal trajectories of the family $y = cx^3$.

Section 2.4

1. Draw a phase line diagram for the equation $y' = (y-1)(y+1)$. Mark the critical points and arrows and describe the stability of the critical points.

2. Draw a phase line diagram for the equation $y' = (y^2-1)(y+2)$. Mark the critical points and arrows and describe the stability of the critical points.

3. Draw a phase line diagram for the equation $y' = y(y-1)(y+1)$. Mark the critical points and arrows and describe the stability of the critical points.

4. Draw a phase line diagram for the equation $y' = y(y-3)(y^2+1)$. Mark the critical points and arrows and describe the stability of the critical points.

5. Draw a phase line diagram for the equation $y' = y^3 - y^2 - 6y$. Mark the critical points and arrows and describe the stability of the critical points.

6. Draw a phase line diagram for the equation $y' = y^3 - 3y^2 - y + 3$. Mark the critical points and arrows and describe the stability of the critical points.

Section 3.1

1. Solve the system

$$\begin{aligned} x' &= -3x, \\ y' &= 2y, \end{aligned}$$

with $x(0) = -1$ and $y(0) = 2$. Hint: Each equation can be solved on its own. Check your solution by substituting in the equations. Does your solution confirm the conclusions of Theorem 3.1.1? Explain!

2. Solve the system

$$\begin{aligned} x' &= 2x, \\ y' &= x - y, \end{aligned}$$

with $x(0) = 3$ and $y(0) = 2$. Hint: Solve one equation and substitute in the other. Check your solution by substituting in the equations. Does your solution confirm the conclusions of Theorem 3.1.1? Explain!

3. Solve the system

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= 2y, \end{aligned}$$

with $x(0) = 1$ and $y(0) = 1$. Hint: Solve one equation and substitute in the other. Check your solution by substituting in the equations. Does your solution confirm the conclusions of Theorem 3.1.1? Explain!

4. Solve the system

$$\begin{aligned} x' &= x^4 y, \\ y' &= y, \end{aligned}$$

with $x(0) = 1$ and $y(0) = 3$. Is this solution defined for all t? Are any solutions defined for all t? Does your solution confirm the conclusions of Theorem 3.1.1. Explain!

5. Solve the system

$$\begin{aligned} x' &= e^t x + 3y, \\ y' &= -y, \end{aligned}$$

with $x(0) = 1$ and $y(0) = 3$.

6. Use the package 2×2System to draw the field of

$$\begin{aligned} x' &= 2x - 3y, \\ y' &= x - 2y \end{aligned}$$

in the rectangle $[-6,6]\times[-4,4]$. Draw orbits for initial conditions $(1,1)$, $(-1,2)$, and $(0,0)$ at $t=0$. Explain what is happening with this last initial condition.

7. Use the package 2×2System to draw the field of

$$\begin{aligned} x' &= x+y-2, \\ y' &= x-y \end{aligned}$$

in the rectangle $[-6,6]\times[-4,4]$. Draw orbits for initial conditions $(-1,3)$, $(-2,2)$, and $(1,1)$ at $t=0$. Explain what is happening with this last initial condition.

8. Describe the relation between the vectors $x'(t)\vec{i}+y'(t)\vec{j}$ and $f(x(t),y(t))\vec{i}+g(x(t),y(t))\vec{j}$ assuming that $\{x(t),y(t)\}$ solves the system

$$\begin{aligned} x' &= f(x,y), \\ y' &= g(x,y). \end{aligned}$$

What is the relation between $x'(t)\vec{i}+y'(t)\vec{j}$ and the normalized field vector at at the point $(x(t),y(t))$.