

# ECE 306 - NATURAL RESPONSES - INVESTIGATION 9 NATURAL RESPONSES OF LINEAR DISCRETE SYSTEMS

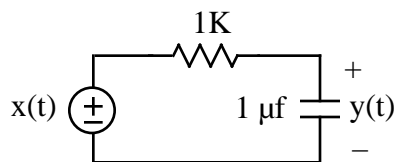
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From our previous Investigations we know how to find the responses of linear difference equations by direct calculation and by convolution. In this Investigation we develop the method of characteristic equations for finding *closed form* equations like the following

$$y_n[n] = 2(0.5)^n u[n]$$

for the natural responses of linear difference equations. We begin in this Investigation with the observation that the step and impulse responses of linear discrete systems look a lot like the step and impulse responses of their analog cousins like the following first order RC circuit we looked at in Investigation 1



This leads us to believe, as we can show, that the responses of linear difference equations can be expressed as a sum of a natural response and a forced response as follows

$$y[n] = \text{Natural Response} + \text{Forced Response}$$

just like the responses of analog circuits can where the

Natural Response = That part of the response that is always there no matter what the input

Forced Response = That part of the response forced on the system by the input

The main objective of this Investigation is to show how the method of *characteristic equations* can be used to find expressions for the natural responses of difference equations. These expressions are useful in analysis and design because they tell us what the natural response is doing and how long it's going to take to decay to zero.

1. The objective of this problem is to show how the method of **characteristic equations** can be used to find the natural responses of linear difference equations like the following

$$y[n] + 0.5y[n - 1] = 2x[n]$$

We present this procedure without proof.

- (1) Set the input  $x[n]$  to zero as follows

$$y[n] + 0.5y[n - 1] = 0$$

- (2) Replace

$$y[n] \text{ by } z^0 = 1, \quad y[n - 1] \text{ by } z^{-1}, \quad y[n - 2] \text{ by } z^{-2}, \dots$$

to obtain the **characteristic equation**

$$1 + 0.5z^{-1} = 0 \quad z + 0.5 = 0$$

- (3) Solve the characteristic equation for the **characteristic roots** as follows

$$z + 0.5 = 0 \quad z = -0.5$$

(4) Use the following Table to obtain the corresponding terms in the natural response

CHARACTERISTIC ROOT	NATURAL RESPONSE
One real root	$Ka^n u[n]$
Two real roots $a$	$K_1 a^n u[n] + K_2 n a^n u[n]$
Complex conjugate roots $re^{\pm j\theta}$	$Kr^n \cos(n\theta + \varphi)u[n]$

where  $K$  and  $\varphi$  depend on the input  $x[n]$  together with difference equation's initial condition. For our example with one real root we have that

$$y_n[n] = K(-0.5)^n u[n]$$

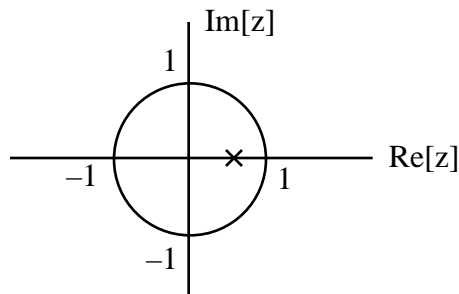
Now use this algorithm to find and sketch the natural responses of the following first order recursive difference equations. Use  $k = 1$  and  $\varphi = 0$  for your sketches.

- $y[n] + 0.7y[n-1] = 5x[n]$
- $y[n] - 0.7y[n-1] = 5x[n]$
- $y[n] + y[n-1] = 5x[n]$
- $y[n] - y[n-1] = 5x[n]$
- $y[n] + 2y[n-1] = 5x[n]$
- $y[n] - 2y[n-1] = 5x[n]$

- Describe how the natural responses of first order recursive difference equations can be different from one another
- From Problems (1) and (2) we see that the natural responses of first order difference equations can
  - Decay to zero
  - Oscillate forever
  - Increase to infinity

Which of these natural responses is not possible in first order linear analog systems

- The objective of this problem is to see how the natural response of a first order difference equation is related to the location of its characteristic root in the complex plane as follows



Note that the circle is referred to as the **unit circle** and we have marked the location of the characteristic root with an **X**. Also note that we refer to this plot as a *characteristic root diagram*. **Memorize** this term. Draw graphs to illustrate what the natural response  $y_n[n]$

looks like when a characteristic root is located on the real line in the characteristic root diagram as follows

- a.  $0 < z < 1$
- b.  $-1 < z < 0$
- c.  $z = 1$
- d.  $z = -1$
- e.  $z > 1$
- f.  $z < -1$

- 5. Summarize in words your results in Problem (4). In particular
  - a. Where are the characteristic roots when the natural response goes to zero
  - b. Where are the characteristic roots when the difference equation is unstable
- 6. Find the characteristic roots of the following second order difference equation

$$y[n] + 0.5y[n - 1] + 0.7y[n - 2] = 2x[n]$$

and then plot them in the characteristic root diagram

- 7. Find and sketch the natural responses of second order difference equations with the following characteristic roots. Show the locations of these characteristic roots in the complex plane. Be careful. In the analog world the characteristic roots correspond to the natural response by

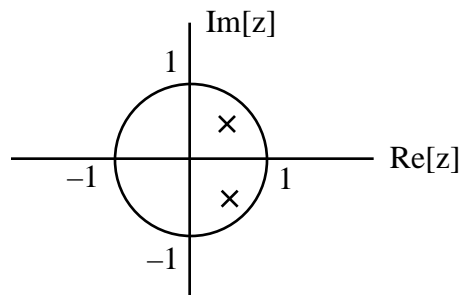
$$a \pm jb \quad y_n(t) = ke^{at} \cos(bt + \varphi)$$

while in the discrete world the characteristic roots correspond to the natural response by

$$a \pm jb = re^{\pm j\theta} \quad y_n[n] = kr^n \cos(n\theta + \varphi)$$

- a.  $0.5 \pm j0.5$
- b.  $0.7 \pm j0.7$
- c.  $1.5 \pm j1.5$

- 8. Make use of your results in Problem (7) to describe how the natural response of a difference equation with all its characteristic roots inside the unit circle differs from the natural response of a difference equation with a characteristic root outside the unit circle. **Memorize** this result. It's very important.
- 9. Suppose we have a pair of complex conjugate characteristic roots inside the unit circle as follows



How does their proximity to the unit circle affect the number of sample times  $n$  it takes the natural response to decay. Draw pictures to illustrate. **Memorize** this result.

10. Math Review: Express  $a = \frac{2}{1 + e^{j1.2}}$  as a complex exponential

11. Math Review: Sketch the distance  $d$  between the point  $a$  and the point  $e^{j\theta}$  in the complex plane as a function of  $\theta$  as  $\theta$  increases from 0 to  $2\pi$

