ECE 306 - DISCRETE CONVOLUTION - INVESTIGATION 7
DISCRETE CONVOLUTION - PART I

WINTER 2007 **A.P. FELZER**

As we've seen it's very straightforward to calculate the responses of discrete systems with difference equations like the following

$$y[n] = 0.5y[n-1] + 2x[n] - 0.8x[n-1] \qquad \text{and even} \qquad y[n] = y^2[n-1] + nx[n]$$

All we have to do is substitute in the numbers for $x[n]$ and do the calculations - even for nonlinear and time-varying systems.

The objective of this and the next investigation is to develop an alternative way to calculate the responses of **linear time-invariant (LTI)** discrete systems as sums of impulse responses. These sums - which we refer to as **convolution sums** - are particularly useful because of the insight they give us into the behavior of linear time-invariant discrete systems.

1. The objective of this first problem is to express a sequence $x[n]$ as a sum of impulses
   a. First sketch a graph of the following sequence $x[n]$

n	$x[n]$
0	2
1	−3
2	1
3	2

   b. Then express $x[n]$ as a sum of impulses $\delta[n],\ \delta[n-1],\ \delta[n-2],\ \ldots$

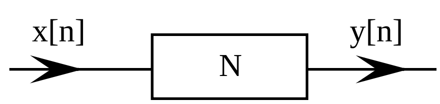
2. Generalizing on the result of Problem (1) we have that any input sequence $x[n]$ as follows

$$x[0],\ x[1],\ x[2],\ \ldots$$

can be expressed as a sum of impulses and delayed impulses as follows

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Now suppose a particular linear time-invariant discrete system as follows



with impulse response $h[n]$ has the input $x[0] = 3,\quad x[1] = -2,\quad x[2] = 1$

   a. First express $x[n]$ as a sum of impulses
   b. Then make use of the fact that N is linear and time-invariant to express the zero state response to $x[n]$ as a sum of impulse responses $h[n],\ h[n-1],\ \ldots$

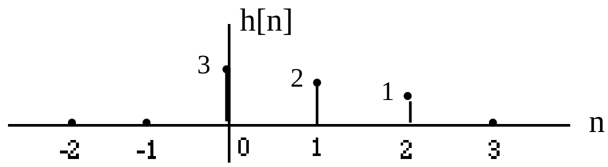
3. Generalizing on the result of Problem (2) we can show that the zero state responses of linear time-invariant discrete systems to inputs $x[n]$ as follows

$$x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

can by **superposition** be expressed as sums of impulse responses as follows

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

where $x[0]h[n]$ is the response to $x[0]\delta[n]$, $x[1]h[n-1]$ is the response to $x[1]\delta[n-1]$ and so on. We call this the **convolution sum**. **Memorize** this equation. Now suppose the impulse response of a particular linear time-invariant difference equation is as follows



and the input is $x[n] = 2\delta[n] + 3\delta[n-1] - \delta[n-2]$

a. Sketch $x[0]h[n] = 2h[n]$ equal to the zero state response to $x[0]\delta[n] = 2\delta[n]$
b. Sketch $x[1]h[n-1] = 3h[n-1]$ equal to the zero state response to $x[1]\delta[n-1] = 3\delta[n-1]$
c. Sketch $x[2]h[n-2] = -h[n-2]$ equal to the zero state response to $x[2]\delta[n-2] = -\delta[n-2]$
d. Now graphically add up your results as follows

$$y[n] = x[0]h[n] + x[1]h[n-1] + x[2]h[n-2]$$

to obtain a sketch of $y[n]$

4. Now suppose we have a linear time-invariant discrete system N with impulse response $h[0] = 3$, $h[1] = 1$, $h[2] = -1$ and all other $h[n] = 0$
   a. Sketch the impulse response $h[n]$
   b. Use the convolution sum to find the zero state response to $x[n] = 2\delta[n] + 3\delta[n-1]$ like you did in Problem (3)

5. The objective of this problem is to get some more practice with convolution sums. Suppose a linear time-invariant discrete system N with the following difference equation

$$y[n] = 2x[n] + 0.5x[n-1] - x[n-2]$$

has the nonzero inputs

$$x[0] = 3, \quad x[1] = -2, \quad x[2] = 2$$

   a. First find the zero-state response to $x[n]$ by direct calculation from the difference equation. Put your result in a Table
   b. Find and plot the impulse response $h[n]$
   c. Express $x[n]$ as a sum of impulses
   d. Make use of the convolution sum to find the zero state response to $x[n]$. Put your result in a Table.
   e. Verify that your convolution result in part (d) is equal to your result in part (a).

6. Generalizing on the results of Problems (4) and (5) we have that the zero state response $y[n]$ of a linear time-invariant discrete system N to the general input

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad \text{is given by} \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

where $h[n]$ is the system's impulse response. We write

y[n]= x[n] h[n]

Memorize the equation for the convolution sum and the corresponding notation.

a. Justify the fact that if N is causal then the convolution sum reduces to

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

b. Now justify the fact that if N is causal and $x[n] = 0$ for $n < 0$ then

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

c. Write out the expression for y[3] of a causal system assuming $x[n] = 0$ for $n < 0$
d. Verify that the sum in part (b) gives the same equation for $y[n]$ as in Problem (4)

7. The objective of this problem is to practice the kind of graphing we'll be doing in the next Investigation. Given that $x[0] = 1.5$, $x[1] = 2$, $x[2] = -0.5$
   a. Sketch $w_1[n] = x[n]$
   b. Sketch $w_2[n] = x[n-1]$
   c. Sketch $w_3[n] = x[-n]$. Then describe how to obtain $x[-n]$ from $x[n]$
   d. Sketch $w_4[n] = x[1-n]$. Then describe how to obtain $x[1-n]$ from $x[n]$

8. Math Review: Find the frequency of

$$x(t) = 3e^{-j2\pi 300t} + 2e^{-j2\pi 100t} + 2 + 2e^{j2\pi 100t} + 3e^{j2\pi 300t}$$