ECE 306 - THE VERY BASICS - INVESTIGATION 6
BASIC PROPERTIES OF LINEAR DISCRETE SYSTEMS

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In the last two Investigations we calculated and explored the responses of recursive and nonrecursive difference equations to discrete steps, impulses and sinusoids. We saw that nonrecursive systems reached steady state values quickly whereas recursive systems may or may not reach steady state depending on the values of the characteristic roots (poles). The objective of this Investigation is to look at some of the basic properties of linear difference equations including linearity, time-invariance, causality and stability.

1. We begin with a review problem
   a. Find and plot the impulse response $h[n]$ of $y[n] = 2x[n] + x[n-1]$
   b. Find and plot the impulse response $h[n]$ of $y[n] = 0.8y[n-1] + x[n]$ for $n = 0,1,\ldots,5$

2. We begin with linearity. We say a difference equation is **linear** if its *zero state responses* - if its responses to inputs $x[n]$ when all initial conditions are zero as follows

$$y[-1] = 0\,,\ y[-2] = 0\,,\ \ . \ . \ .$$

   satisfy the following:

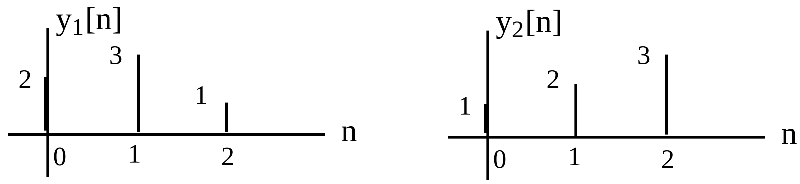
   (1) If $y[n]$ is the zero state response to $x[n]$ then the zero state response to $ax[n]$ is $ay[n]$ as follows

   (2) More generally if $y_1[n]$ and $y_2[n]$ are the zero state responses to $x_1[n]$ and $x_2[n]$ as follows

   then the zero state response to the sum $x_1[n] + x_2[n]$ is equal to the sum of the responses $y_1[n] + y_2[n]$ as follows

   (3) And more generally still the zero state response to the more general sum $ax_1[n] + bx_2[n]$ is $ay_1[n] + by_2[n]$

Memorize these relations for linearity. Then assuming that the discrete signals $y_1[n]$ and $y_2[n]$ as follows



are the zero state responses of a given DSP to the inputs $x_1[n]$ and $x_2[n]$

a. Sketch the zero state response to $x[n] = 2x_1[n]$ if the DSP is linear
b. Sketch a possible zero state response to $x[n] = 2x_1[n]$ if the DSP is not linear
c. Sketch the zero state response to $x[n] = x_1[n] + x_2[n]$ if the DSP is linear
d. Sketch the zero state response to $x[n] = x_1[n] + x_2[n]$ if the DSP is not linear

3. Show that linearity is satisfied for the following nonrecursive difference equation

$$y[n] = x[n] - 3x[n-1]$$

for the input $x[n] = 2x_1[n] - 3x_2[n]$ for when

n	$x_1[n]$	$x_2[n]$
-1	0	0
0	1	2
1	2	-1
2	-1	1

To do this you need to

(1) Find $y_1[n]$ equal to the zero state response to $x_1[n]$
(2) Find $y_2[n]$ equal to the zero state response to $x_2[n]$
(3) Find $x[n] = 2x_1[n] - 3x_2[n]$
(4) Find $y[n]$ equal to the zero state response to $x[n]$
(5) And then verify that $y[n]$ as calculated in step (4) is equal to $y[n] = 2y_1[n] - 3y_2[n]$

Be sure to put your results in a Table

4. Generalizing on the results of Problem (3) it can be shown that all nonrecursive difference equations like

$$y[n] = x[n] - 3x[n-1]$$

and all recursive difference equations like

$$y[n] = y[n-1] + x[n] - 3x[n-1]$$

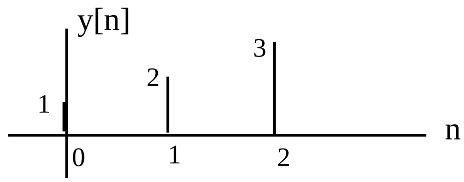
are linear. The objective of this problem is to look at some examples of nonlinear systems.

a. Show that $y[n] = x^2[n]$ is not linear for the inputs $x_1[n]$ and $x_2[n]$ from Problem (3). In particular show that the response to $x[n] = x_1[n] + x_2[n]$ is not $y_1[n] + y_2[n]$
b. Come up with an example of your own of a nonlinear discrete difference equation

5. The objective of this problem is to define what we mean by the time-invariance of discrete systems. We say a discrete system as follows

is **time-invariant** if a delay in the input simply causes a delay in the zero state response $y[n]$ as follows

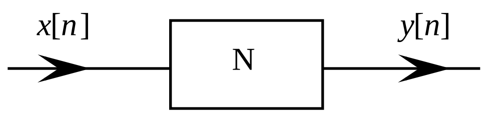
Memorize this definition. Now suppose that the discrete signal y[n] as follows



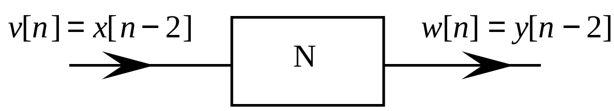
is the zero state response to the input $x[n]$

a. Sketch the zero state response to $x[n-2]$ if the DSP is time-invariant
b. Sketch a possible zero state response to $x[n-2]$ if the DSP is not time-invariant

6. Demonstrate that $y[n] = x[n] - 3x[n-1]$ is time-invariant for an input $x[n]$ of your choice. In particular show that if $y[n]$ is the zero state response to $x[n]$ as follows



then the zero state response to $v[n] = x[n-2]$ will be equal to $w[n] = y[n-2]$ as follows



Put your results in a Table

7. Show that the DSP characterized by the discrete equation

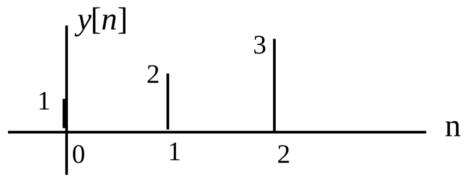
$$y[n] = nx[n]$$

is not time-invariant. Hint - compare its responses to the unit step $u[n]$ and the delayed unit step $u[n-2]$

8. Find a difference equation for a DSP that is linear but not time-invariant.

9. How can you tell from the coefficients of a difference equation if it's time-invariant.

10. Suppose the zero state response to $x[n]$ of a linear time-invartiant difference equation is as follows

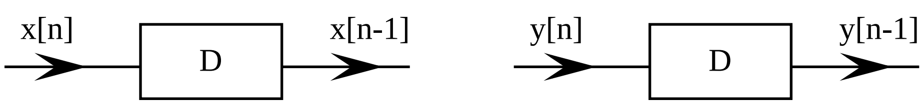


Find the zero state response to $2x[n]+x[n-1]$

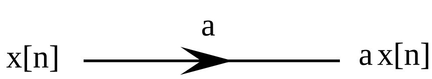
11. The objective of this problem is causality. A discrete system is said to be **causal** if the output y[n] does not depend on future inputs. **Memorize** this definition. Which of the following discrete systems are causal and which not. Explain how you know
    a. $y[n]=y[n-1]+x[n]-3x[n-1]$
    b. $y[n]=y[n-1]+x[n+1]-3x[n]$
    c. $y[n]=2x^2[n]$

12. Find a difference equation for a discrete system that is neither linear, causal or time-invariant.

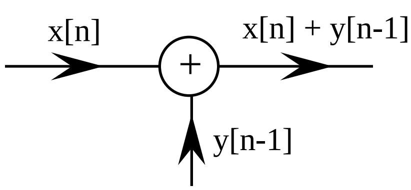
13. The objective of this problem is to introduce how difference equations are implemented in hardware. The main components of such implementations are as follows:
    (i) Registers (D Flip Flops) to store previous values of x[n] and y[n] as follows



(ii) Multipliers

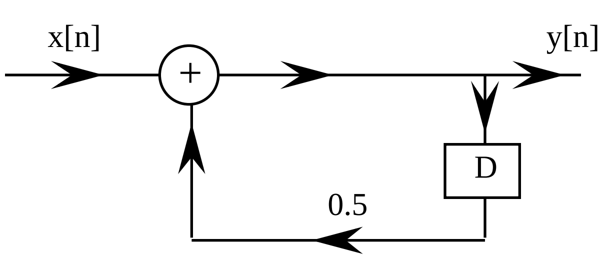


(iii) Adders

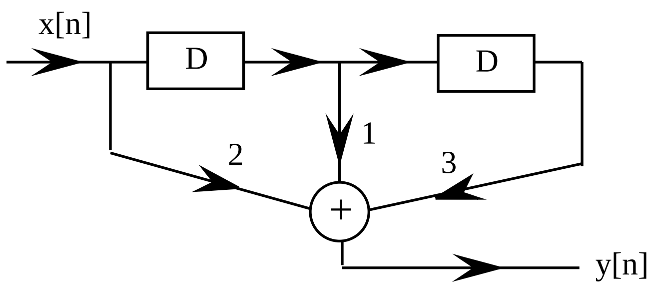


Now there are, as you might guess, a large number of different ways to realize difference equations just like there are innumerable ways to realize analog circuits and systems. What follows are some typical examples. In each case find the difference equation for y[n].

a.



b.



14. Math Review: Express $x(t) = 2\cos(2\pi 1000t) - 3\cos(2\pi 2000t + 1.2)$ as a sum of complex exponentials