ECE 306 - THE VERY BASICS - INVESTIGATION 4
SAMPLING - PART II

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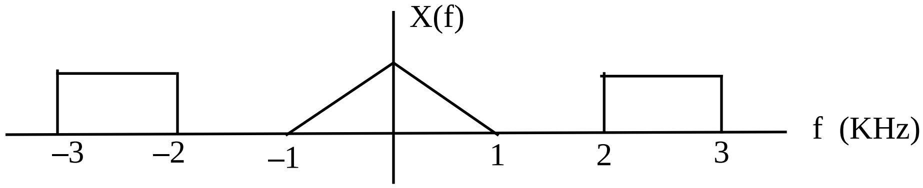
The main result of the last Investigation is the Sampling Theorem which tells us that we can reconstruct a bandlimited signal $x(t)$ from its samples $x[n] = x(nT_s)$ if the sampling frequency f_s is fast enough - if it satisfies

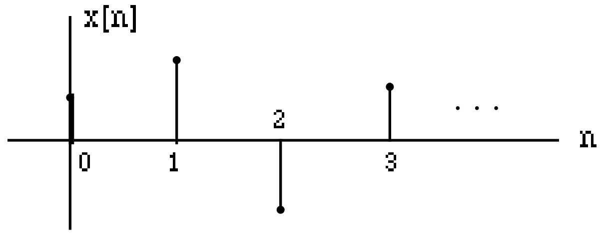
$$f_s > 2f_b$$

The main objectives of this Investigation are to show how this reconstruction can be done and to see what happens when we don't sample fast enough.

1. We begin with a review problem. How fast must the following bandlimited signal be sampled in order to reconstruct it from its samples



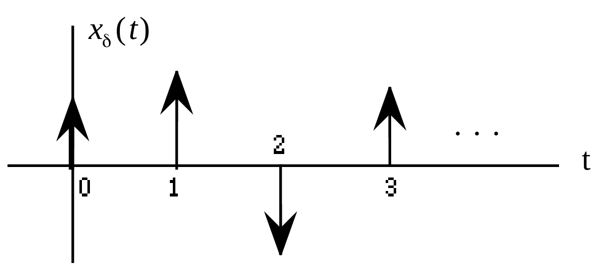
2. The objective of this problem is to introduce the key observation that a discrete signal like the following



is *equivalent* to the following continuous impulse sampled signal

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

obtained by multiplying every sample value by an impulse as follows



a. Find and sketch the impulse sampled signal for the samples

$$x[0] = 1,\ x[1] = 2,\ x[2] = -3,\ x[3] = 0,\ x[4] = 1$$

b. Find the samples $x[-1],\ x[0],\ .\ .\ .\ ,\ x[3]$ for the following impulse sampled signal

$$x_\delta(t) = \delta\left(t + T_s\right) + 3\delta(t) + \delta\left(t - T_s\right) - 2\delta\left(t - 2T_s\right) + 2\delta\left(t - 3T_s\right)$$

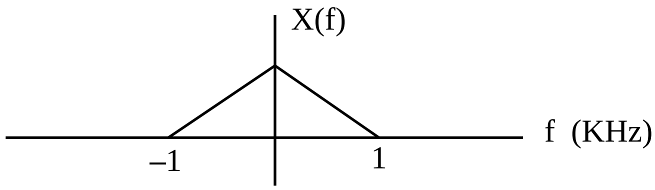
if $T_s = 1$ msec

c. What do we mean when we say that the sequence of samples $x[n]$ and the impulse sampled signal $x_\delta(t)$ are equivalent

3. The key result for impulse sampled functions is that their spectrums are equal to the following sum

$$X_\delta(f) = F\left[x_\delta(t)\right] = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X\left(f - kf_s\right)$$

The objective of this problem is to use this result to show how we can, at least ideally, reconstruct a bandlimited signal from its samples. Suppose in particulart that a bandlimited signal $x(t)$ has the following spectrum



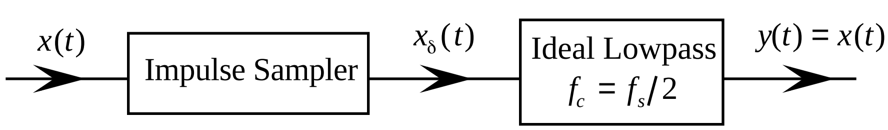
a. Sketch $X_\delta(f)$ if $f_s = 3000$ samples/sec

b. Sketch the spectrum $Y_\delta(f)$ at the output of the following ideal lowpass filter

with cutoff frequency $f_c = \dfrac{f_s}{2} = 1500\ Hz$

c. Make use of your result in part (b) to show that $y_\delta(t) = x(t)$. Hint - compare the spectrums of $y_\delta(t)$ and $x(t)$

4. From Problem (3) we know that we can, at least ideally, reconstruct a bandlimited signal $x(t)$ from its samples as follows

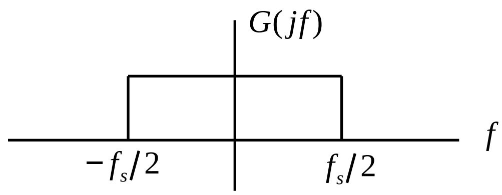


by doing the following

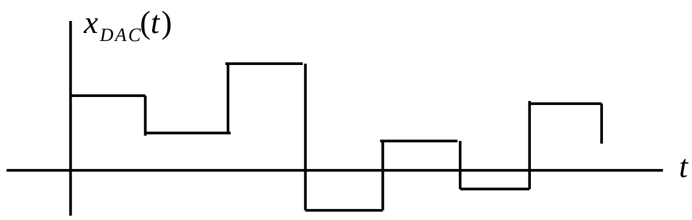
(1) Sample $f_s > 2f_b$

(2) Multiply every sample by an impulse to create the continuous signal $x_\delta(t)$

(3) Pass $x_\delta(t)$ through an ideal lowpass filter with the following frequency response



Now in real digital signal processing circuits we don't of course multiply our samples $x[n]$ by impulses. Instead we put them through a circuit called a **digital-to-analog converter (DAC)** that converts them to a staircase signal just like the one from the sample-and-hold as follows



and then pass this signal through a lowpass filter to get a signal reasonably close to $x(t)$

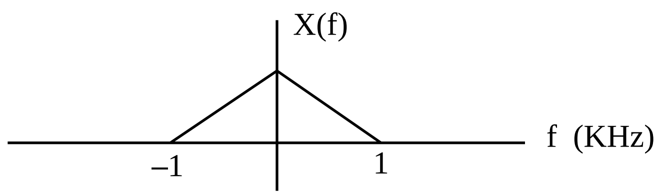
a. Sketch $x_{DAC}(t)$ for $x[0]=1$, $x[1]=2$, $x[2]=-3$, $x[3]=0$, $x[4]=1$
b. Now sketch $x_{DAC}(t)$ after it goes through a lowpass filter
c. What's the purpose of the lowpass filter. Note that it's possible to design this lowpass filter to obtain $x(t)$ exactly

5. We know from the Sampling Theorem that there is only one signal $x(t)$ bandlimited by $f_b < f_s/2$ that has a given set of samples. And that we can get it back from its samples - at least in principle - as follows

   (1) Multiply $x(t)$ by an impulse train

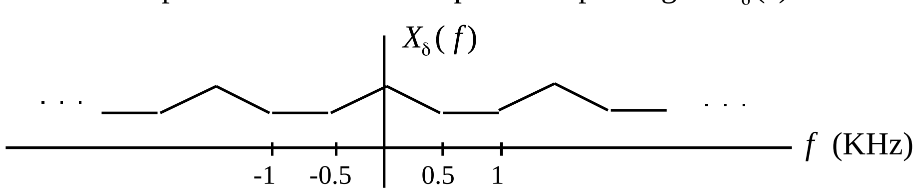
   (2) Put $x_\delta(t)$ through an ideal lowpass filter

   The objective of this problem is to see what happens when we try to recover $x(t)$ in this same way when we're not sampling fast enough - when we're sampling at a frequency $f_s \le 2f_b$. Suppose in particular that we sample the signal $x(t)$ with the following spectrum



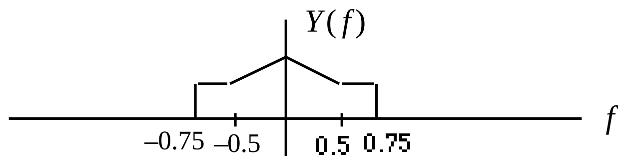
at the frequency f_s = 1500 samples/sec

a. First show that the spectrum of the ideal impulse sampled signal $x_\delta(t)$ looks as follows

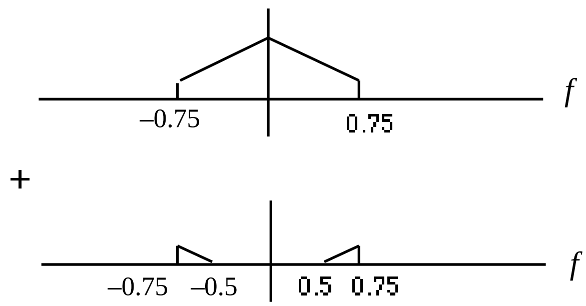


b. Then find and sketch the spectrum of $y(t)$ at the output of the ideal lowpass filter as follows with cutoff frequency $f_c = f_s/2 = 0.75$

c. Now from part (b) we know that the spectrum at the output of the ideal lowpass filter as follows



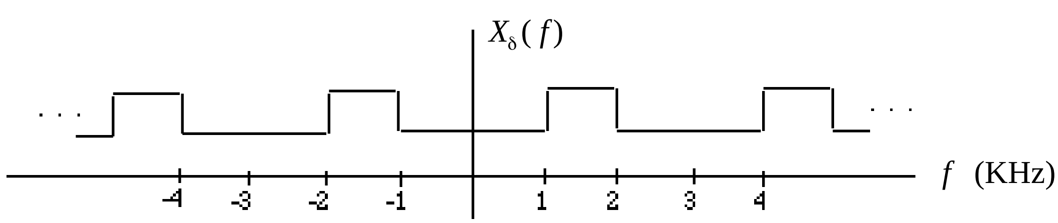
is equal to the following sum



Why would you expect we refer to $f_s/2 = 0.75$ as the **folding frequency**. Hint - explain how $Y(f)$ can be constructed from a single $X(f)$

d. How can we tell from part (b) that $x(t) \neq y(t)$

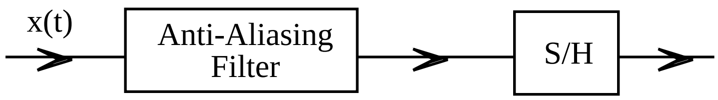
6. From Problem (5) we see that we can't reconstruct a bandlimited signal from its samples when $f_s < 2f_b$. We refer to the resulting distortion of y(t) as **aliasing**. **Memorize** this term. We get aliasing because the samples of bandlimited signals are not all different when $f_s < 2f_b$. Demonstrate this fact by finding the spectrums of two different signals $x_1(t)$ and $x_2(t)$ bandlimited by $f_b = 2$ KHz that both have the same impulse sampled spectrum $X_\delta(f)$ as follows when sampled at $f_s = 3$ KHz



7. In many situations we're interested in a signal in a given frequency range like

$$0 < f < f_b$$

but not interested in what's going on at higher frequencies $f > f_b$. To prevent the higher frequency "interference" from affecting the values of the samples we first put $x(t)$ through a lowpass **anti-aliasing filter** before it reaches the sample-and-hold as follows



a. Sketch the frequency response of an ideal anti-aliasing filter if we're only interested in that part of the signal in the frequency range

$$0 < f < 1500$$

b. What would you choose for f_s. Justify your answer

8. Math Review: Given Euler's Relation as follows

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

a. Find $z = 3e^{j2.1}$ in rectangular form - as a sum of a real and an imaginary part
b. Plot $z = 3e^{j2.1}$ in the complex plan
c. How far is $z = 3e^{j2.1}$ from the origin
d. Express $z = 2 + j3$ in complex exponential form