ECE 306 - THE VERY BASICS - INVESTIGATION 3
SAMPLING - PART I

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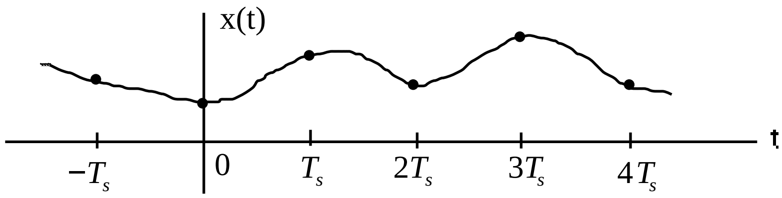
As we saw in Investigation 1, the inputs of discrete systems are sequences of numbers like the following

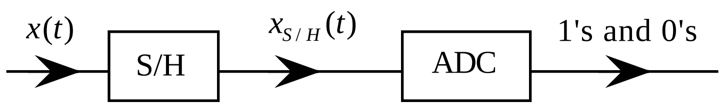
$$x[0] = 0,\ x[1] = 1,\ x[2] = -1,\ \ldots$$

obtained by sampling continuous signals $x(t)$ like the following

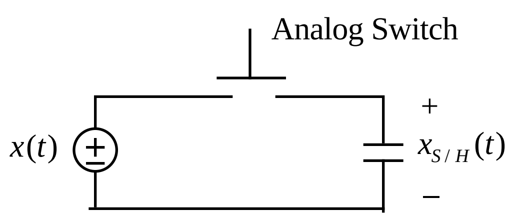


The main objective of this and the next Investigation is to see how fast we have to sample such signals to be able to recreate them from their samples $x[n] = x(nT_s)$.

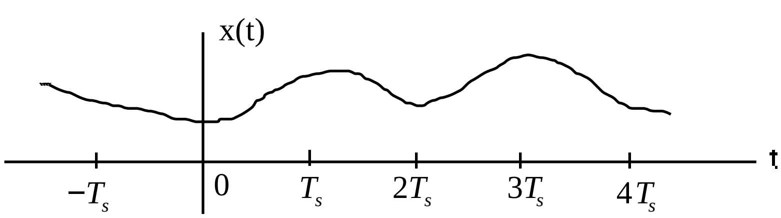
1. Sampling is done with a circuit called a **Sample and Hold (S/H)** followed by an **Analog-to-Digital Converter (ADC)** as follows



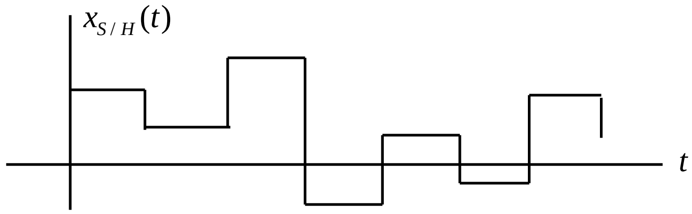
(1) The S/H circuit is a circuit like the following



with an analog input like



and a staircase output $x_{S/H}(t)$ like the following



that *holds* the sample values $x[n] = x(nT_s)$

(2) The Analog-To-Digital Converter - really a souped up digital voltmeter - is a circuit that converts the sample value $x[n] = x(nT_s)$ to binary

Note that when the **sampling interval $\mathbf{T_s}$** - the time between samples - is a constant then we say the sampling is **uniform sampling**.

a. Explain how our simple sample-and-hold circuit works - how it's able to sample and then hold $x(t)$
b. Calculate and then sketch $x_{S/H}(t)$ as shown above for two cycles together with a timing diagram for the voltage V_{sw} that opens and closes the analog switch assuming that a high voltage closes the switch and a low voltage opens it

2. Given a signal $x(t) = 2\cos(2\pi 1000t)$
   a. Sketch $x(t)$
   b. Sketch $x_{S/H}(t)$ starting at time $t = 0$ if $T_s = 0.2$ msec
   c. Calculate the first ten samples $x[n] = x(nT_s)$ of $x(t)$ starting at time $t = 0$ if $T_s = 0.15$ msec. Put your results in a Table
   d. Make a discrete plot of $x[n]$ versus n for your results in part (c)

3. What would you expect we mean by the **sampling frequency $\mathbf{f_S}$**. What are its units. Come up with an equation for f_s as a function of the sampling interval T_s.

4. Find and put in a Table the first 5 samples of x(t) = 3 cos (2π100t) if the sampling frequency f_s = 500 samples/sec

5. Based on the results of the last three problems we see that sampling is conceptually, at least, pretty straightforward. Reconstructing a signal from its samples, however, takes a little more thought. Let's begin by supposing we have the following 4 samples

   x[0] = 1   x[1] = 2   x[2] = 2   x[3] = 1

   Draw three different curves that go through these sample points.

6. Generalizing on the result of Problem (5) we see that there are an infinite number of signals going through any given set of sample values. The objective of this problem is to look at the special case of sinusoids.
   a. Show that the first 5 samples of

   $$x_1(t) = 5\cos(2\pi 5t) \quad \text{and} \quad x_2(t) = 5\cos(2\pi(5 + f_s)t)$$

   are the same when we sample at the rate f_s = 15 samples/sec.
   b. Generalize on your results from part (a) to show that all the samples of $x_1(t)$ and $x_2(t)$

are equal when f_s = 15 samples/sec. In particular show that

$$x_1[n] = x_1(nT_s) = 5\cos(2\pi 5 nT_s) \quad \text{and} \quad x_2[n] = x_2(nT_s) = 5\cos(2\pi(5 + f_s)nT_s)$$

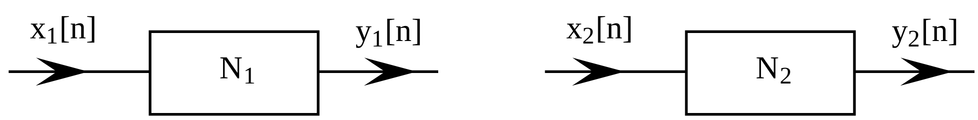
are equal for all n. Hint - multiply out the terms in $x_2[n]$

c. And finally generalize on part (b) to show that all the samples $x_1[n] = x_1(nT_s)$ and $x_2[n] = x_2(nT_s)$ of

$$x_1(t) = 5\cos(2\pi f_1 t) \quad \text{and} \quad x_2(t) = 5\cos(2\pi(f_1 + mf_s)t)$$

are equal for all integers m. **Memorize** this result forever.

7. Now suppose we have two identical Digital Signal Processors N_1 and N_2 as follows

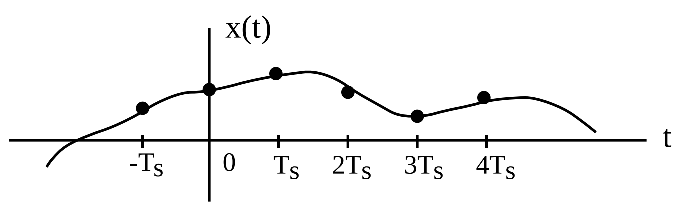


with $x_1[n]$ and $x_2[n]$ equal to the samples of the following cosines

$$x_1[n] = x_1(nT_s) = 5\cos(2\pi f_1 nT_s) \quad \text{and} \quad x_2[n] = x_2(nT_s) = 5\cos(2\pi f_2 nT_s)$$

with $f_2 = f_1 + mf_s$. Then what is the relation between $y_1[n]$ and $y_2[n]$. How do you know.

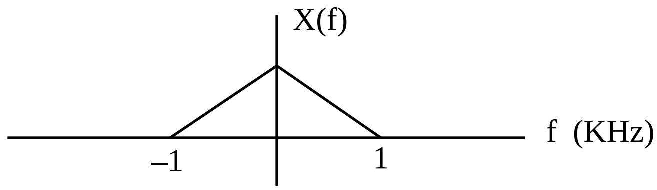
8. From Problems (5) and (6) we know that every set of samples x[n] has an infinite number of signals x(t) going through them. And so it *looks like* we can't reconstruct $x(t)$ from its samples. On the other hand, it seems reasonable that if we take a particular signal like the following



and sample it "fast enough" then we should be able to get a pretty good approximation of $x(t)$ by simply "connecting the dots". The question is how fast is fast enough. To answer this question let us review what we mean by bandlimited signals. A signal $x(t)$ with spectrum $X(f)$ is bandlimited by f_b if

$$X(f) = 0 \quad \text{for all } f \text{ such that } |f| > f_b$$

a. Sketch the spectrum $X(f)$ of a signal $x(t)$ that is bandlimited by f_b = 2000 Hz
b. What is f_b for $x(t) = 3 + 2\cos(2\pi 100t) + 4\cos(2\pi 300t)$
c. What is f_b for $x(t)$ with the following spectrum



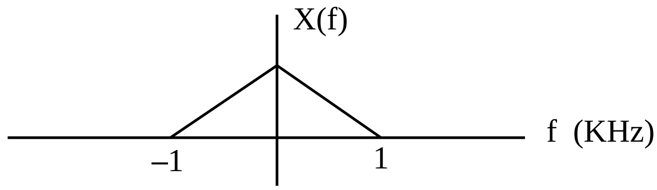
9. Now that we've defined what we mean by bandlimited signals we can state the **Sampling Theorem** as follows:

   If a signal $x(t)$ is **bandlimited** by the frequency f_b then $x(t)$ can be reconstructed from its samples $x[n]$ if the **sampling frequency** $\mathbf{f_s > 2f_b}$. Note that we refer to the frequency $\mathbf{2f_b}$ of a bandlimited signal as its **Nyquist Frequency**.

   Memorize the Sampling Theorem forever. It tells us that the samples of bandlimited signals are like their fingerprints. They'll always be different as long as $f_s > 2f_b$.

   How fast do we need to sample each of the following signals to be able to reconstruct it from its samples

   a. $x(t) = 3 + 2\cos(2\pi 100t) + 4\cos(2\pi 300t)$
   b. $x(t)$ with the spectrum



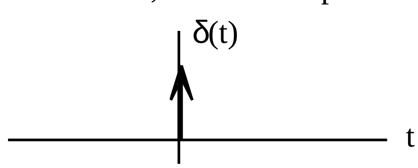
10. What's wrong with sampling $x(t) = 2\sin(2\pi 1000t)$ at the frequency $f_s = 2000$ samples/sec. Hint - how are the samples of $x(t)$ and $w(t) = -2\sin(2\pi 1000t)$ related when $f_s = 2000$ samples/sec

11. Review of Difference Equations: Calculate and plot the output of a discrete system with difference equation

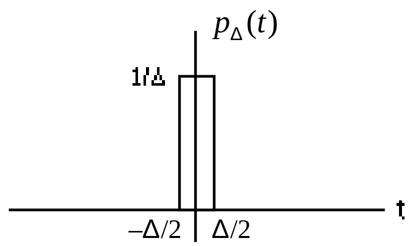
$$y[n] = 0.7y[n-1] + x[n] \qquad \text{for } n = 0, 1, 2, 3, 4$$

   if $x[n] = 1$ for all $n \geq 0$ and $y[-1] = 0$.

12. Review of Impulses: Impulse functions, which we represent symbolically as follows



   are the "limits" as Δ goes to zero of unit pulses as follows



What characterizes impulse functions is that even though they're zero everywhere except at $t = 0$, they still manage to have areas equal to one. Impulses are certainly not functions in the "regular" sense of high school math and beginning calculus. Nevertheless they are bona fide mathematical entities that turn out to be very useful in simplifying the mathematics of circuit and system analysis. What makes them relatively straightforward to work with is that they behave just like very, very narrow pulses of area one. Note that the *impulse response* of a circuit or system is the response when the input is an impulse and all initial conditions are equal to zero. Given all this

a. Sketch $\delta(t-2)$
b. Sketch $\delta(t-2)+2\delta(t+2)$
c. Sketch $2t\delta(t-2)$
d. Make use of your result in part (c) to find $\int_{-\infty}^{\infty} 2t\delta(t+2)dt$