

ECE 306 - THE Z-TRANSFORM - INVESTIGATION 22

SOLVING DIFFERENCE EQUATIONS WITH THE Z-TRANSFORM

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To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we introduced the z-transform and then calculated it for some common signals. In particular we found that

$$Z[2|K|a^n \cos(bn + K)] = \frac{Kz}{z - ae^{jb}} + \frac{K^*z}{z - ae^{-jb}}$$

The objective of this Investigation is to use our results to find the complete responses of linear difference equations with constant coefficients to common signals like steps $u[n]$.

1. We begin with the following difference equation

$$y[n] = 0.5y[n-1] + \delta[n] \quad y[-1] = 0$$

- a. First find $Y(z)$ by taking the z-transform of both sides of the equation
- b. Make use of your Table of z-transforms from the last Investigation to find $y[n]$
- c. Sketch $y[n]$

2. As we saw in the last problem, it's very straightforward to solve difference equations like the following

$$y[n] = 0.5y[n-1] + \delta[n]$$

All we have to do is go to our Table and get the corresponding $y[n]$. But for more general z-transforms like the following ratio of polynomials in z

$$Y(z) = \frac{z^2}{(z + 0.5)(z - 0.7)}$$

we need to make use of the result from algebra that such a rational polynomial can be written as a sum of terms as follows

$$Y(z) = \frac{z^2}{(z + 0.5)(z - 0.7)} = K_0 + \frac{K_1z}{z + 0.5} + \frac{K_2z}{z - 0.7}$$

with constants K_0 , K_1 and K_2 . Note that these partial fraction expansions are very similar to the ones we use for LaPlace Transforms. Now there are several ways to obtain the coefficients K_0 , K_1 and K_2 . Go through the calculations for the following algorithm:

- a. Find K_0 by simply evaluating both sides of the equation with $z = 0$
- b. Find K_1 by multiplying both sides of the equation for $Y(z)$ by

$$\frac{z + 0.5}{z}$$

and setting $z = -0.5$. What happens to all the terms on the right hand side of the

- equation except the one for K_1
 c. From part (b) we have that

$$K_1 = \frac{z + 0.5}{z} Y(z) \Big|_{z=-0.5}$$

Make use of this scheme to find K_2

- d. Make use of your results in part (a)-(c) to express $Y(z)$ as a sum of fractions and then make use of your Table of z-transforms in the previous Investigation to find an expression for $y[n]$
3. Find and sketch the closed form solution of the **step response** - the response when
- (1) The input is the unit step: $x[n] = u[n]$
 - (2) All initial conditions are zero

of the difference equation

$$y[n] = -0.7y[n-1] + 2x[n]$$

4. Find the closed form solution of the impulse response - the response when $x[n] = \delta[n]$ and all initial conditions are zero - of the following difference equation

$$y[n] = -0.5y[n-2] + 1.5y[n-1] + 2x[n]$$

5. Find the closed form solution of the impulse response of the the following difference equation

$$y[n] = 1.3y[n-2] + 1.5y[n-1] + 2x[n]$$

6. Find the closed form solution of the zero state response of the following difference equation with $x[n] = \cos(3n)u[n]$

$$y[n] = 0.5y[n-1] + 2x[n]$$

7. Make use of the fact that if

$$Y(z) = z^{-1}W(z) \quad \text{then} \quad y[n] = w[n-1]$$

to find $y[n]$ when

$$Y(z) = \frac{3}{z-0.5}$$

Hint - note that $Y(z) = z^{-1} \frac{3z}{z-0.5}$. Check your answer by evaluating the difference equation for $y[n]$

8. Find the closed form solution of the impulse response of the following difference equation when $x[n] = \delta[n]$

$$y[n] = 0.5y[n-1] + 2x[n-1]$$