ECE 306 - FOURIER ANALYSIS - INVESTIGATION 20
INTRODUCTION TO THE FFT

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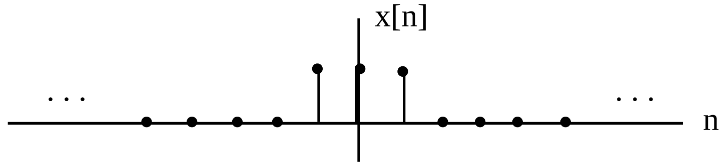
To do "well" on this investigation you must not only get the right answers but must also do neat, complete and concise writeups that make obvious what each problem is, how you're solving the problem and what your answer is. You also need to include drawings of all circuits as well as appropriate graphs and tables.

In the last Investigation we introduced the DFT and IDFT and developed some of their basic properties. We showed, in particular, that

(1) The DFT is linear
(2) When $x[n]$ is nonzero for only a finite number of samples like the following discrete pulse



then the DFT is equal to the Fourier Transform X(Ω) at the frequencies k(2π/N) as follows

$$\mathbf{X}[k] = \mathbf{X}\left(k\left(\frac{2\pi}{N}\right)\right)$$

Now for a computer or special purpose digital signal processor to directly calculate an N-point DFT or IDFT as follows

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n} \qquad \text{and} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{jk(2\pi/N)n}$$

it must do N^2 complex multiplications as well as a like number of additions. The time it takes for these calculations basically depends on the number of multiplications since they take much longer than the additions. And how long it takes to do the multiplications depends in general on the number of bits in the factors and whether fixed point or floating point arithmetic is being used.

The objective of this investigation is to introduce how the FFT - Fast Fourier Transform - takes advantage of the symmetries and periodicity of the complex exponential

$$e^{j2\pi/N}$$

to more efficiently calculate the DFT. Note that chips designed to do DSP are very often simply microprocessors fine tuned to speed up the operations used most often in digital signal processing - especially multiplication.

1. Show that it takes $N^2 = 4^2 = 16$ complex multiplications to calculate a 4-point DFT - that is, to calculate the DFT of four samples $x[0], x[1], x[2]$ and $x[3]$

2. When we work with the DFT and FFT we usually let

$$W_N = e^{-j2\pi/N}$$

Write the equations for the DFT and IDFT in terms of W_N

3. There are a number of different variations on the FFT algorithm. In this investigation we will be doing what is referred to as the *decimation in time FFT*. Suppose in particular that we have the following 4-point DFT

$$X[k] = \sum_{n=0}^{3} x[n] W_N^{kn}$$

   a. First show that $X[k]$ can be written as follows

   $$X[k] = G[k] + W_4^k H[k]$$

   where $G[k] = x[0]W_4^0 + x[2]W_4^{2k}$ and $H[k] = x[1]W_4^0 + x[3]W_4^{2k}$

   b. Now show that $G[k]$ in part (a) is equal to the following 2-point DFT

   $$G[k] = x[0]W_4^0 + x[2]W_4^{2k} = \sum_{n=0}^{1} x_e[n] W_2^{kn}$$

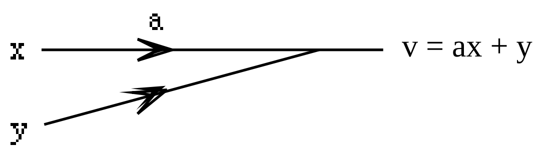
   of the even number samples $x_e[n] = x[2n]$ for $n = 0,1$. Hint - Write out the terms in the summation and then equate terms.

   c. And then show that H[k] in part (a) is equal to the following 2-point DFT

   $$H[k] = x[1]W_4^0 + x[3]W_4^{2k} = \sum_{n=0}^{1} x_o[n] W_2^{kn}$$

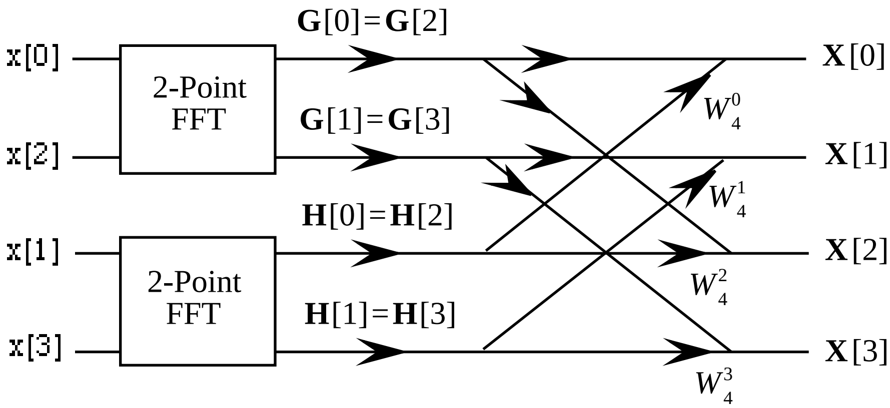
   of the odd number samples $x_o[n] = x[2n+1]$ for $n = 0,1$

4. The objective of this problem is to draw a diagram representing the 4-point decimation in time FFT we calculated in Problem (3). But first note that when we draw diagrams as follows

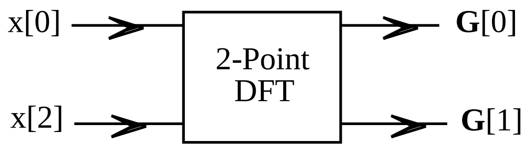


the arrows tell us to multiply x by a and y by 1 and the lines coming together tell us to add the corresponding terms to obtain $v = ax + y$.

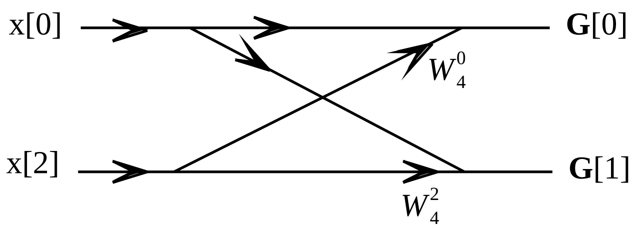
Using such diagrams we can represent our 4-point decimation in time FFT as follows



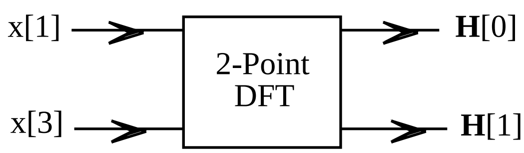
where the DFT for $x[0]$ and $x[2]$ represented by



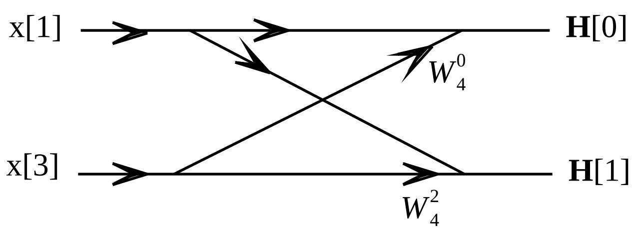
is given by



and the DFT for $x[1]$ and $x[3]$ represented by



is given by



Note that we refer to the basic calculations like those for G and H as **butterflies**.

Now suppose we have

$$x[0]=1 \qquad x[1]=-2 \qquad x[2]=1 \qquad x[3]=2$$

a. First calculate the 4-point DFT directly by hand or with Mathcad or Matlab.
b. Now draw the complete diagram for the 4-point decimation in time FFT with all the lines and arrows.

c. And then calculate the 4-point decimation in time FFT by first calculating the 2-point DFT's $G[k]$ and $H[k]$ as follows

$$G[k] = x[0]W_4^0 + x[2]W_4^{2k} = \sum_{k=0}^{1} x_e[n]W_2^{kn}$$

$$H[k] = x[1]W_4^0 + x[3]W_4^{2k} = \sum_{k=0}^{1} x_o[n]W_2^{kn}$$

and then using them to calculate the $X[k]$'s in your diagram in part (b)

d. Verify that your results in parts (a) and (c) give the same results.

e. How many separate multiplications by W_N^k were required to calculate the FFT in part (c)

5. Generalizing on the result of Problem (4) it can be shown that as a result of the periodicity of G[k], H[k] and their cousins in higher order FFT's, it only takes on the order of

$$N \log_2(N)$$

complex multiplications to calculate the DFT using the decimation in time FFT instead of N^2 by direct calculation. How many times faster will a digital processor be able to calculate an $N = 2^{10} =$ 1024-point DFT with the FFT than by direct calculation.

6. Math Review: Show that $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ when $|r| < 1$