ECE 306 - THE VERY BASICS - INVESTIGATION 1
INTRODUCTION TO DISCRETE SIGNALS AND SYSTEMS

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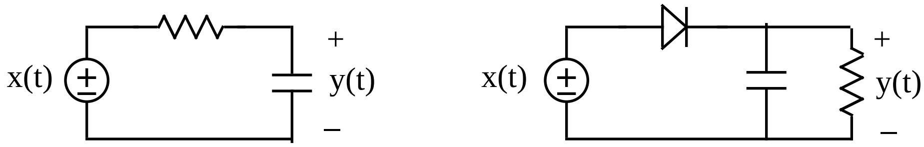
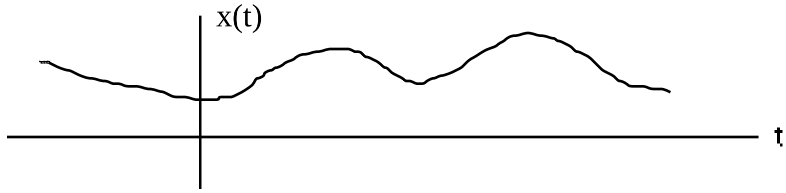
From our previous classes we know how to build *analog* circuits like the following



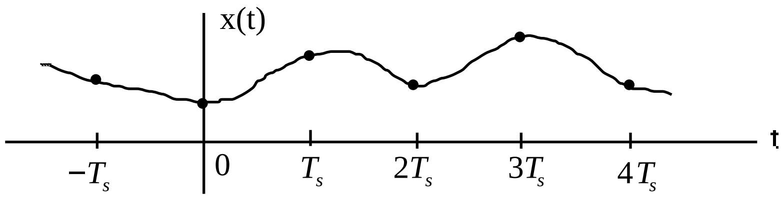
that can attentuate, amplify, filter, rectify and so on *continuous* inputs $x(t)$ like the following



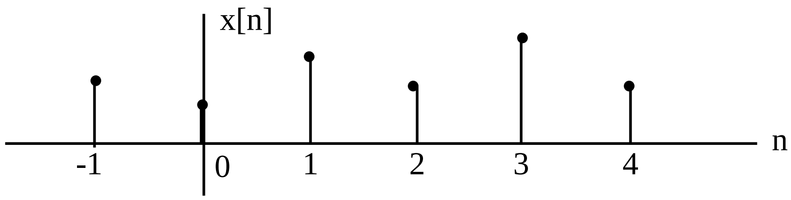
Our main objective in this class is to show *discrete* systems of the following form

can attenuate, amplify, filter, rectify and so on *discrete* signals $x[n]$. In such systems

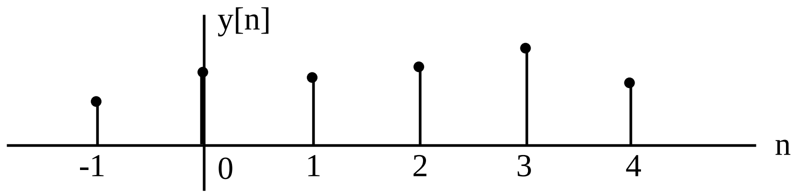
(1) The analog inputs $x(t)$ are *sampled* every T_s seconds as follows



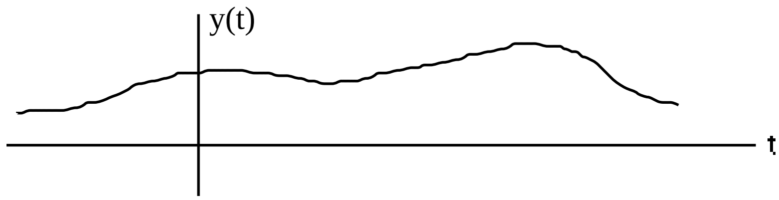
to obtain the *discrete sample values* $x[n] = x(nT_s)$ as follows



(2) The sample values - the numbers $x[n] = x(nT_s)$ - are then added, subtracted, multiplied and so on by a digital circuit made from gates and flip-flops (really a souped up calculator) to give us discrete signals $y[n]$ like the following

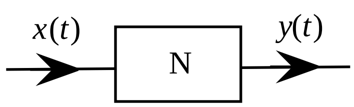


(3) And finally the discrete outputs $y[n]$ of the digital circuit are converted back to analog to give us continuous outputs $y(t)$ like the following



Our main objective in this first Investigation is to introduce the difference equations of discrete systems and show how they're related to the differential equations of analog circuits.

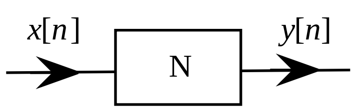
1. We begin with a review problem. Given an analog system N as follows



with $x(t) = 5\cos(2\pi 1000t)$

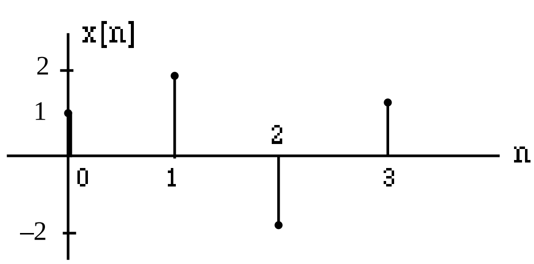
   a. Sketch $x(t)$
   b. Sketch $y(t)$ if N is an attenuator with $y(t) = 0.5x(t)$
   c. Sketch $y(t)$ if N is an amplifier with $y(t) = 2x(t)$
   d. Sketch $y(t)$ if N is a rectifier with $y(t) = |x(t)|$

2. Now suppose that N is a discrete system with input $x[n]$ and output $y[n]$ as follows



with $x[-2] = 1,\ x[-1] = 1,\ x[0] = -1,\ x[1] = -2,\ x[2] = 1,\ x[3] = 1$

   a. Sketch a *discrete* plot like the following



   of $x[n]$ as a function of n

   b. Sketch $y[n]$ as a function of n if N is an attenuator with $y[n] = 0.5x[n]$
   c. Sketch $y[n]$ as a function of n if N is an amplifier with $y[n] = 2x[n]$

d. Sketch $y[n]$ as a function of n if N is a rectifier with $y[n] = |x[n]|$

3. Algebraic equations like $y[n] = 0.5x[n]$ and $y[n] = 2x[n]$ for discrete attenuators and amplifiers are examples of simple **difference equations**. The goal of this and the next two problems is to use methods from numerical analysis to obtain more general difference equations for circuits containing capacitors and inductors. But before we actually do this let us consider the following related problem.

   Suppose that after three and one half hours on the road we've driven $x(3.5) = 230$ miles

   a. What's the total distance $x(4)$ after four hours if in the last half hour we drive a constant 70 mph
   b. What's the total distance $x(4)$ after four hours if the average speed in the last half hour is 60 mph
   c. Generalizing on part (b) we have that

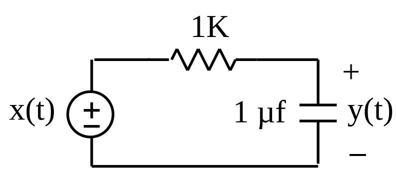
   $$x(4) = x(3.5) + 0.5\left(\text{Average speed during the last half hour}\right)$$

   But speed is the derivative of distance. So we can write

   $$x(4) = x(3.5) + 0.5\left(\text{Average value of } \frac{dx(t)}{dt} \text{ during the last half hour}\right)$$

   Now in real problems we usually don't know the average value of the derivative and so we have to make an estimate. The simplest estimate - especially if we're on an open road - is to approximate our average speed in the last half hour by the speed at the beginning of the interval. We call this **Euler's method**. Use Euler's method to approximate $x(4)$ if the speed at the beginning of the last half hour is 64 mph

4. The objective of this problem is to apply our results on Euler's method to obtain a difference equation for the following RC circuit



   a. First verify that the differential equation for $y(t)$ is as follows

   $$\frac{dy(t)}{dt} + 1000y(t) = 1000x(t)$$

   Hint - write the node equation at the output

   b. We now apply Euler approximation to obtain a difference equation for the voltage $y(t)$ across the capacitor at time $t = t_0$. We begin like we did in the last problem

   $$y(t_o) = y(t_o - \Delta t) + \Delta t \begin{pmatrix} \text{Average rate at which } y(t) \text{ is changing} \\ \text{during the interval } t_o - \Delta t \le t \le t_o \end{pmatrix}$$

   $$y(t_o) = y(t_o - \Delta t) + \Delta t \left(\text{Average of } \frac{dy(t)}{dt} \text{ during } t_o - \Delta t \le t \le t_o\right)$$

   where

$$y(t_0 - \Delta t) = \text{the voltage across the capacitor at time } t = t_0 - \Delta t$$

And then make use of Euler's Method to approximate the average of the derivative by the value of the derivative at the beginning of the interval as follows

$$\left(\text{Average of } \frac{dy(t)}{dt} \text{ during } t_o - \Delta t \le t \le t_o\right) \approx \frac{dy}{dt}(t_o - \Delta t)$$

Doing this we obtain the following general expression for approximating $y(t_0)$

$$y(t_o) \approx y(t_o - \Delta t) + \Delta t \frac{dy}{dt}(t_o - \Delta t)$$

So for our differential equation as follows

$$\dot{y} + 1000y = 1000x \qquad \dot{y} = -1000y + 1000x$$

we have

$$\frac{dy}{dt}(t_o - \Delta t) = -1000y(t_o - \Delta t) + 1000x(t_o - \Delta t)$$

and so

$$y(t_o) \approx y(t_o - \Delta t) + \Delta t \frac{dy}{dt}(t_o - \Delta t) = y(t_o - \Delta t) + \Delta t\left[-1000y(t_o - \Delta t) + 1000x(t_o - \Delta t)\right]$$

If we now apply Euler's method at the time $t_o = n\Delta t$ we obtain

$$y(n\Delta t) \approx y((n-1)\Delta t) + \Delta t\left[-1000y((n-1)\Delta t) + 1000x((n-1)\Delta t)\right]$$

Now make use of the following substitutions

$$y[n] = y(n\Delta t) \qquad y[n-1] = y((n-1)\Delta t) \qquad x[n-1] = x((n-1)\Delta t)$$

to verify that the difference equation for $y[n]$ is

$$y[n] - 0.9y[n-1] = 0.1x[n-1]$$

when $\Delta t = 10^{-4}$ sec.

c. Will the response of the difference equation be exactly equal to the response of the original RC circuit at the sampling times. How can you tell
d. How can the approximation be made better

5. We call the result from Problem (4) as follows

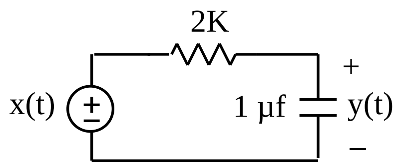
$$y[n] - 0.9y[n-1] = 0.1x[n-1]$$

a **linear difference equation with constant coefficients**. They're the equations for *dynamic discrete systems* just like differential equations are the equations of analog dynamic circuits like our RC circuit. We will be spending a large part of this course working with these equations.

a. How are the difference equations for RC circuits different from those for amplifiers and attenuators

b. Make up your own example of a linear difference equation with constant coefficients

6. Make use of Euler's method to find the difference equation for the following RC circuit



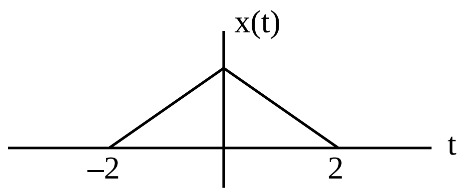
7. Calculate and plot the output of a discrete system with difference equation

$$y[n] = -0.5y[n-1] + x[n] \qquad \text{for } n = 0, 1, 2, 3, 4$$

if $x[n] = 1$ for all $n \geq 0$ and $y[-1] = 0$. Put your results in a Table. Describe in words what's happening to $y[n]$ as n increases.

8. Given all that we've done so far
   a. What do you think are the advantages of discrete systems
   b. What do you think are the limitations of discrete systems

9. Math Review - Given $x(t)$ as follows



   a. Sketch $x(t-2)$
   b. Sketch $x(t+2)$
   c. Sketch $x(t-5) + x(t) + x(t+5)$
   d. Sketch $x(t-3) + x(t) + x(t+3)$

10. Given $x(t) = 5\cos(2\pi 1000t)$
    a. What is the amplitude of $x(t)$
    b. What is the frequency of $x(t)$ in Hz
    c. What is the period of $x(t)$
    d. Sketch $x(t)$

11. Write the equation for a sinusoidal of amplitude 10 and period 10^{-4} sec