ECE 257 - LESSON 21
NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

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IN CLASS

From the last Lesson on integration and differentiation we know that if we know the speed of an object as follows

$$v(t) = 5\cos(2\pi 1000t)$$

then we can approximate the distance the object has travelled at time $t = n\Delta t$ by

$$x(n\Delta t) = x\big((n-1)\Delta t\big) + \Delta t \text{ (Average speed in the } n'th \text{ interval)}$$

$$\approx x\big((n-1)\Delta t\big) + \Delta t \text{ (Speed at the beginning of the } n'th \text{ interval)}$$

$$\approx x\big((n-1)\Delta t\big) + \Delta t \; v\big((n-1)\Delta t\big)$$

And since the speed of an object is equal to the derivative of its speed we have

$$x(n\Delta t) \approx x\big((n-1)\Delta t\big) + \Delta t \; x'\big((n-1)\Delta t\big)$$

The objective of this Lesson is to show that differential equations like

$$x' = -1000x + 5\cos(2\pi 1000t) \qquad x(0) = 1$$

where $x'(t)$ depends not only on a function of t but also $x(t)$ itself can also be solved using numerical methods just like those we used to calculate integrals

1. The objective of this problem is to review the closed form solution of the following differential equation

   $$x' = -1000x + 2000 \qquad x(0) = 1$$

   a. Write out in words what this differential equation says about the derivative of x
   b. Verify that $x(t) = 2 - e^{-1000t}$ satisfies the initial condition $x(0) = 1$
   c. Verify that $x(t) = 2 - e^{-1000t}$ satisfies the differential equation $x' + 1000x = 2000$
   d. Sketch $x(t) = 2 - e^{-1000t}$ for $0 \le t \le 5$ msec
   e. Make use of Matlab to plot $x(t) = 2 - e^{-1000t}$ for $0 \le t \le 5$ msec

2. In Problem (1) we reviewed the nice closed form solution $x(t) = 2 - e^{-1000t}$ of the differential equation

   $$x' = -1000x + 2000 \qquad x(0) = 1$$

   More generally we have to use numerical methods to approximate the solutions of more complicated equations. The key idea is that the solution of any first order differential equation satisfies the following relation we reviewed above for approximating integrals

   $$x(n\Delta t) = x\big((n-1)\Delta t\big) + \Delta t \left(\begin{array}{l}\text{Average rate at which } x(t) \text{ is changing} \\ \text{during the interval } (n-1)\Delta t \le t \le n\Delta t\end{array}\right)$$

But the average rate at which $x(t)$ is changing is the average rate of its derivative. So we have

$$x(n\Delta t) = x\big((n-1)\Delta t\big) + \Delta t \left(\text{Average of } \frac{dx(t)}{dt} \text{ during } (n-1)\Delta t \le t \le n\Delta t\right)$$

Since we don't know the average of the derivative we have to make an approximation. The simplest approximation - as we reviewed above - is Euler's approximation as follows

$$\left(\text{Average of } \frac{dx(t)}{dt} \text{ during } (n-1)\Delta t \le t \le n\Delta t\right) \approx \frac{dx}{dt}\big((n-1)\Delta t\big)$$

From this we obtain the following general expression for approximating $x\big(n\Delta t\big)$

$$x(n\Delta t) \approx x\big((n-1)\Delta t\big) + \Delta t \frac{dx}{dt}\big((n-1)\Delta t\big)$$

So for our differential equation as follows

$$x' = -1000x + 2000 \qquad x(0) = 1$$

we have

$$x(n\Delta t) \approx x\big((n-1)\Delta t\big) + \Delta t \frac{dx}{dt}\big((n-1)\Delta t\big) = x\big((n-1)\Delta t\big) + \Delta t\big[-1000\big((n-1)\Delta t\big) + 2000\big]$$

And so we have

$$x\big(n\Delta t\big) \approx x\big((n-1)\Delta t\big) + \Delta t\big[-1000x\big((n-1)\Delta t\big) + 2000\big]$$

a. Make use of this equation for $x\big(n\Delta t\big)$ to approximate $x(t)$ by hand at $t = \Delta t, 2\Delta t, 3\Delta t$ for $\Delta t = 0.1$ msec
b. Make use of Matlab to approximate $x(t)$ at $t = \Delta t,\ 2\Delta t,\ 3\Delta t$ for $\Delta t = 0.1$ msec
c. Verify that your results in parts (a) and (b) are the same
d. Make use of Matlab to approximate and plot $x(t)$ for $0 \le t \le 5$ msec for $\Delta t = 0.1$ msec. Use a for loop just like in the last Lesson
e. Repeat part (d) for $\Delta t = 0.01$ msec
f. Compare your graphs in parts (d) and (e) with each other and then with your plot of the exact solution $x(t) = 2 - e^{-1000t}$ in Problem (1)