ECE 257 - LESSON 20
INTEGRATION AND DIFFERENTIATION

SPRING 2007 **A.P. FELZER**

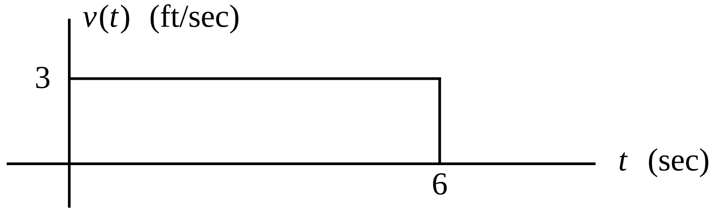
IN CLASS

The objective of this Lesson is to see how Matlab can be used to approximate the integrals and derivatives of functions. In particular we'll see how to approximate

(1) Distance traveled from speed by approximating the integral of the speed

(2) Speed from distance traveled by approximating the derivative of the distance

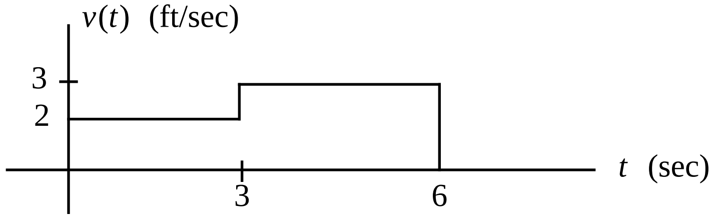
1. We begin with some simple examples of calculating distance from speed. Suppose the speed of an object is constant over an interval as follows



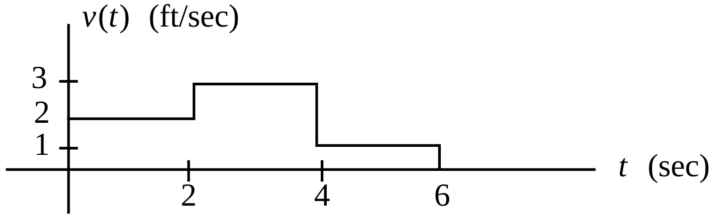
   a. Sketch the speed $v(t)$ for $0 \le t \le 6$ sec
   b. Sketch the distance $x(t) = 3t$ for $0 \le t \le 6$ sec

2. From Problem (1) we see that when the speed is constant the distance traveled increases (or decreases) linearly. Make use of this fact to sketch the distances for the more general cases below.

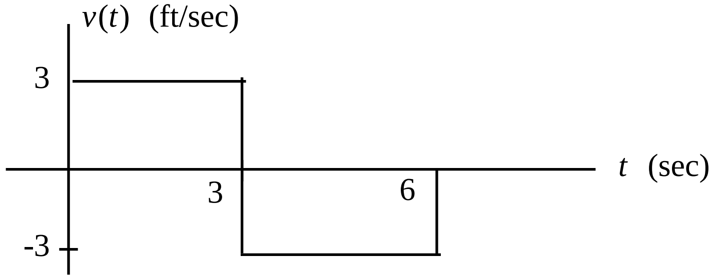
   a.



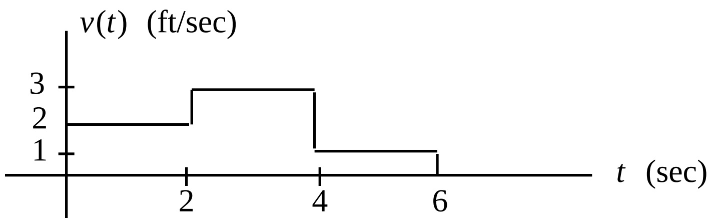
   b.



   c.



3. The objective of this problem is to use Matlab to calculate and plot the distance traveled by an object with the following velocity



Hint -

(1) First create a vector v for the speed in each two second interval

(2) Then make use of a for loop to calculate a distance vector x with

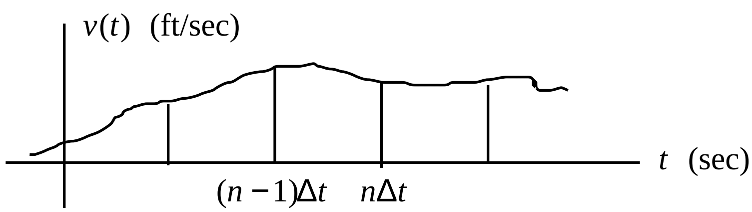
$$x(n) = \text{Distance traveled from } t = 0 \text{ to the end of the } n'th \text{ interval}$$
$$= (\text{Distance traveled from } t = 0 \text{ to the end of the } (n-1)st \text{ interval})$$
$$+ \; \Delta t (\text{Speed in the } n'th \text{ interval})$$

$$x(n) = x(n-1) + (\Delta t) v(n)$$

(3) Plot the distance traveled from $t = 0$ sec to $t = 6$ sec

4. The distances calculated in the previous problems - the areas under the curves - were easy to find because the speed in each interval was constant. More generally, however, the speed varies continuously like in the following graph

Now we can still divide time into intervals of length Δt as follows



but we have to modify our equation for $x(n\Delta t)$ as follows

$$x(n\Delta t) = \text{Distance traveled from } t = 0 \text{ to } t = n\Delta t$$
$$= (\text{Distance traveled from } t = 0 \text{ to } t = (n-1)\Delta t\,)$$
$$+ \; \Delta t \; (\text{Average speed in the } n'th \text{ interval})$$
$$x(n\Delta t) = x\big((n-1)\Delta t\big) + \Delta t \; (\text{Average speed in the } n'th \text{ interval})$$

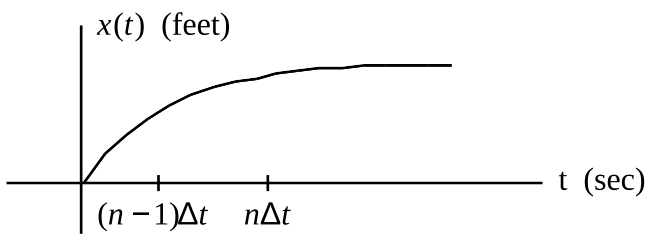
The problem is that in general we don't know the average speed and so the best we can do is approximate it. The simplest approximation of the average speed in an interval is the speed at the beginning of the interval. The rationale for this approximation is that if the intervals are small enough then as the speed goes up and down from one interval to the next the estimate of the total distance traveled after a "long enough" time should be pretty close to the actual value. So we have

$$x(n\Delta t) \approx x\big((n-1)\Delta t\big) + \Delta t \text{ (Speed at the beginning of the } n'th \text{ interval)}$$

Now suppose we have an object traveling at the speed $v(t) = 4t$ ft/sec for $0 \le t \le 5$ sec.

a. Use Matlab to plot the speed $v(t)$
b. Use Matlab to calculate and plot $x(t)$ using the above approximation for $x(n\Delta t)$ with $\Delta t = 1$ sec
c. Repeat part (b) with Δt = 0.1 sec
d. Repeat part (b) with Δt = 0.01 sec
e. Use Matlab to plot the actual distance traveled $x(t) = 2t^2$
f. Compare the above approximations to the actual distance traveled

5. From the previous problems we see how Matlab can be used to approximate the distance from the speed by approximating the area under the curve - by taking its integral. The objective of this and the next problems is to now make use of Matlab to approximate the speed from the distance. We begin by calculating some average speeds. Suppose an object has traveled a distance $x(t)$ as follows



a. Write an equation for its average speed in the time interval $(n-1)\Delta t \le t \le n\Delta t$ as a function of $x(t)$
b. What is the relationship between the slope of the line connecting the points $x\big((n-1)\Delta t\big)$ and $x(n\Delta t)$ and the average speed
c. Find the average speed between $(n-1)\Delta t = 100$ sec and $n\Delta t = 102$ sec if $x(100) = 250$ ft and $x(102) = 272$ ft

6. Suppose our object is going back and forth like a pendulum with $x(t) = \cos(2t)$ feet during the time $0 \le t \le 5$ sec
   a. Use Matlab to obtain a graph of $x(t)$
   b. Use Matlab to calculate the average speed of the car as follows

$$\text{Average Speed} = \frac{x\big((n-1)\Delta t\big) - x(n\Delta t)}{(n-1)\Delta t - n\Delta t} = \frac{\Delta x(t)}{\Delta t}$$

   in each interval of length $\Delta t = 0.5$ sec for $t = 0$ to $t = 5$ sec. And then plot these average speeds as a staircase graph from $t = 0$ to $t = 5$ sec. Hint - make use of a for loop or the diff function
   c. Repeat part (b) for $\Delta t = 0.1$ sec
   d. Repeat part (b) for $\Delta t = 0.01$ sec
   e. Now plot the actual speed $v(t) = -2\sin(2t)$
   f. How does the size of Δt affect the approximation of the speed

7. In the above problems we showed how to use Matlab to approximate the speed as the derivative

of the distance and the distance as the integral of the speed as follows

$$v(t) = \frac{d}{dt}x(t) \quad \text{and} \quad x(t) = \int v(t)dt$$

Putting these two results together we have

$$x(t) = \int v(t)dt = \int \left(\frac{d}{dt}x(t)\right)dt \qquad\qquad x(t) = \int \left(\frac{d}{dt}x(t)\right)dt$$

which, with some restrictions, is true for any $x(t)$. What do we call this result