What happens when a*t > c?

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Three questions

Q1: Is there any theoretical limit on your acceleration?

Q2: Is there any theoretical limit on how long a time you can accelerate?

Q3: Is there any theoretical limit on your speed?

So what's up with *that?*

One more question

Q4: Taking only human physiology into account, how do we maximize the distance traveled by a single individual who starts from rest on Earth?

A4: Accelerate constantly with the *maximum* acceleration, which is probably pretty close to

$$g \approx 1\, c/\text{year} = 1 \text{ lightyear/year}^2 \equiv \text{“}g\text{”}$$

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Aside: The Newtonian result

- A 100 year trip at constant, maximum acceleration reaches a final speed of

  $$v = at \approx 100\ \mathrm{c}$$

  and covers a distance of

  $$d = \frac{1}{2}at^2 \approx 5{,}000 \text{ light years}$$

- Compare that result with
  our galaxy's size: ~100,000 light years
  the distance to the next galaxy: ~2,000,000 light years
  the size of the universe: $\sim 10^{10}$ light years (?)

- Yes, pretty pathetic, but at least the distance traveled doubles with only an additional 41% of whatever time has already been spent.

  :-ʃ

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What do you mean by “your speed”?

- Implies the existence of an observer with a privileged status.
- Often refers to a reference frame that *is* privileged in a very practical sense—*e.g.*, when the speed is being measured in a reference frame with respect to which you are *usually* at rest.
- What may matter most is how long *you* say it takes for you to move some distance as measured by the *privileged* observer (“PO”). This distinction is irrelevant classically but is, of course, critical relativistically.

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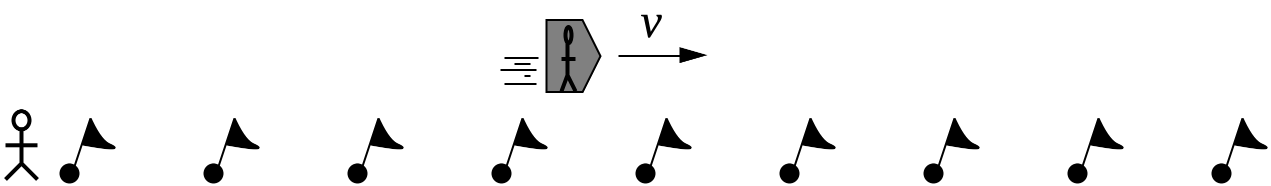
"Apparent Speed"

In light of the foregoing ...

- Consider a line of markers that are equally spaced with separation D in their own rest frame—the frame of the PO. The traveler moves past the markers with a speed v and observes the markers to be passing with frequency f .



- Define the "apparent speed" as $v_{\text{app}} = fD$ or, more formally,

$$v_{\text{app}} \equiv \frac{dx_{\text{priv}}}{dt_{\text{trav}}} =$$ Rate at which the traveler's position as measured by the PO changes with respect to time as measured by the traveler.

(Hereafter $x_{\text{priv}} = x$ and $t_{\text{trav}} = t$.)

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The relationship between apparent speed v_{app} and observed speed v

- The frequency $f = \dfrac{v}{D/\gamma} = \dfrac{\gamma\ v}{D}$ so that $v_{\text{app}} = \gamma\, v$.

- Other useful relationships:

$$v = \frac{v_{\text{app}}}{\sqrt{1 + (v_{\text{app}}/c)^2}} \text{ so } \gamma \equiv \frac{1}{\sqrt{1 - (v/c)^2}} = \sqrt{1 + (v_{\text{app}}/c)^2}$$

$$d_{\text{app}} \equiv \oint v_{\text{app}} dt = \Delta x$$

$$a_{\text{app}} \equiv \frac{dv_{\text{app}}}{dt} = \gamma^3 \frac{dv}{dt}$$

$$t_{\text{priv}} = \oint \gamma\ dt$$

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What do you mean by "your acceleration"?

- What *you* mean is surely what *you feel*—*i.e.*, what an accelerometer carried with you would read. This is the rate at which your speed changes with respect to instantaneously comoving observers (ICO's). Call this the "local acceleration" and indicate its central status by denoting it simply as *a*.
- In a time *dt* your speed (wrt to the ICO's) changes by an amount *a dt*. Your "observed speed" (with respect to the PO's) changes by an amount

$$dv = \frac{v + adt}{1 + v\,adt/c^2} - v = adt\left[1 - (v/c)^2\right]$$

$$\frac{dv}{dt} = a\left[1 - (v/c)^2\right] = a/\gamma^2 \quad (\quad a_{app} = \gamma a)$$

which can be integrated to obtain

$$v = c\,\tanh\left\{\frac{1}{c}\int a dt + \frac{1}{2}\ln\left[\frac{1 + v_0/c}{1 - v_0/c}\right]\right\}$$

A journey at constant local acceleration

- If the local acceleration is constant and the traveler begins at rest at t = 0 we find

for $t >> c/a$

$$v = c\tanh\left(\frac{at}{c}\right) \approx c\left\{1 - 2\exp\left(\frac{-2at}{c}\right)\right\}$$

so that

$$v_{\text{app}} = \gamma v = c\sinh\left(\frac{at}{c}\right) \approx \frac{c}{2}\exp\left(\frac{at}{c}\right)$$

Yielding

$$d_{\text{app}} = x = \frac{c^2}{\text{a}}\left[\cosh\left(\frac{at}{c}\right) - 1\right] \approx \frac{c^2}{2\text{a}}\exp\left(\frac{at}{c}\right)$$

$$a_{\text{app}} = a\cosh\left(\frac{at}{c}\right) \approx \frac{a}{2}\exp\left(\frac{at}{c}\right)$$

Finally

$$t_{\text{priv}} = \int \gamma\, dt = \frac{c}{a}\sinh\left(\frac{at}{c}\right) = \frac{v_{\text{app}}}{a} \approx \frac{c}{2a}\exp\left(\frac{at}{c}\right)$$

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A trip with a = “g” = 1 c/year

t (yr)	t_{priv} (yr)	v (c)	v_{app} (c)	a_{app} (“g”)	d_{app} (c yr)
1	1.18	0.762	1.18	1.54	0.543
2	3.63	0.964	3.63	3.76	2.76
4	27.3	0.99933	27.3	27.3	26.3
10	1.1×10^{4}	$1 - 4 \times 10^{-9}$	1.1×10^{4}	1.1×10^{4}	1.1×10^{4}
20	2.4×10^{8}	$1 - 8 \times 10^{-18}$	2.4×10^{8}	2.4×10^{8}	2.4×10^{8}
25	3.6×10^{10}	$1 - 4 \times 10^{-22}$	3.6×10^{10}	3.6×10^{10}	3.6×10^{10}

Doubling time = $\frac{c}{a} \ln 2 \approx 8$ months (effective after a few years)

Shortly after year 20 cosmological effects become dominant

Things I find interesting about all of this

1. Dispelling misconceptions about the “limitations” of relativistic spacetime: Is there a “speed of light barrier”?
2. Recognizing the “human physiology barrier”: $a_{\text{max}} \approx g$
3. The surprising limitations to travel in Newtonian spacetime.
4. The exponential characteristics of constant acceleration travel in relativistic spacetime: The 8 month doubling time.
5. The intrusion of cosmological effects even on a relatively short and perfectly comfortable trip.

Another Resource

Phil Fraundorf, University of Missouri-St. Louis

http://www.umsl.edu/~fraundor/