

Appendix C: Analysis of Uncertainty

In an experiment, we make direct measurements of quantities like lengths, times, masses, etc. The instruments used to make these measurements vary in quality as does the expertise of the researchers using them. Furthermore, parameters that are out of the control of the researcher may cause undesired variations in the things that are being measured. Often, the very thing being measured may not even *have* a well-defined value like, for example, the diameter of a cylinder that isn't perfectly round (as is *always* the case to some degree).

These measured values are used in calculations to obtain experimental results that are usually compared with the predictions of theory or with the results of other experiments. Since experimental results are based on measurements that are *unavoidably* uncertain, there will inevitably be *some* difference between the values being compared. Thus, the important question arises, "Is the difference more than could reasonably be expected *given* the uncertainties in the measured values?" Put more simply we mean, "How likely is it that the difference is *real*?"

"Analysis of uncertainty" (often misleadingly called "*error* analysis") is a three step process that consists of 1) estimating the unavoidable uncertainty in one's experimental measurements, 2) determining the resulting uncertainty in the calculated results, and 3) using those uncertainties to evaluate the significance of experimental discrepancies.

As you read the following sections and as you perform the experiments in this course please keep these points in mind:

- 1 Estimates of uncertainty are just that—estimates. For our purposes they are *intended* to represent "worst case" values—i.e., your estimate of the largest amount by which the *true* value of a quantity might differ from the *measured* value. Although estimates of uncertainties are never "guesses," reasonable people may assign uncertainties to the same measurement that vary by a factor of two or three. Keeping this in mind can significantly simplify the analysis of uncertainty and improve your ability to evaluate your results.
- 2 *Genuine* discrepancies between experiment and theory *do* occur even in student labs. These are generally due to *real* effects that have not been taken into account in the theory used to analyze the data. One of the most interesting and challenging aspects of data analysis is identifying possible reasons for such genuine discrepancies.
- 3 Notwithstanding the above, if you are reasonably careful in the lab, you will *rarely* observe discrepancies larger than 20 or 30%. Whenever you *do* obtain discrepancies that large it is *quite* likely that you have made a mistake either in collecting your data or in performing the required calculations. Please recognize the *enormous* difference between a discrepancy of 10% and one of a *factor* of ten. The former may arise as a result of unavoidable uncertainties in the data; the latter is almost *certainly* the result of a mistake.

We will use the following notation and terminology in all that follows:

Let “ x ” be the name of a measured or calculated quantity. Then

x_{true} = the “*true value* of ‘ x .’” (Usually unknown and unknowable!)

x_{best} = our “*best estimate* of x_{true} ”

x = the “*absolute uncertainty* in x_{best} ,” i.e., the largest likely value of $|x_{\text{best}} - x_{\text{true}}|$. (Often called simply *the uncertainty*.)

$\frac{x}{x_{\text{best}}}$ = the “*relative or fractional uncertainty* in x_{best} ,” often expressed as a percentage and then called the “*percentage uncertainty* in x_{best} .”

x_{min} = $x_{\text{best}} - x$ = the *minimum* likely value of x_{true} .

x_{max} = $x_{\text{best}} + x$ = the *maximum* likely value of x_{true} .

x x_{min} to x_{max} = $x_{\text{best}} \pm x$ = our “*experimental range* for ‘ x ’” (Usually just called the “*experimental value* of ‘ x .’”)

Uncertainties in Measurements

Estimating the uncertainty in measurements is the first step in the analysis of uncertainties and it *must* be done *at the time that the measurements are made*; it can not be done correctly as an afterthought.

Sources of uncertainty may be grouped into two categories

Uncertainty due to instrumental precision: For our purposes the instrumental precision can be determined simply from the scale of the instrument and how accurately you can read it. (More properly it also includes random effects that occur in the *operation* of the instrument and also some understanding of how well the instrument is *calibrated*.) Instrumental precision sets a *lower limit* on the uncertainty in a measurement. Please note carefully, however, that it is *quite* rare for the uncertainty in a measurement to be determined by this factor. One of the most common mistakes made by students in the introductory lab is *underestimating* the uncertainty in a measurement by assuming that it is equal to the instrumental precision and ignoring the generally larger “sample uncertainty.”

Sample uncertainty: Sample uncertainty is determined by the “inherent variability” in the measurement being made. For instance, if one is timing the fall of a ball with a digital stopwatch the sample uncertainty is likely to be *larger* than the instrumental precision because of the difficulty of insuring that it is always dropped from the same height, that it isn’t given a slight push, that you start and stop the stopwatch at the right times, etc. Measuring the width of a table with a meter stick would probably have less sample uncertainty unless, of course, the width actually *does* vary along its length. Sample uncertainty also results from the role of the researcher’s judgment in using the instrument.

It is worth taking a moment to identify two types of *errors* that may afflict experimental measurements. We have *neither* of these in mind when we discuss experimental *uncertainty*.

Mistakes, blunders, etc.: These come in *many* forms and include such lapses of technique as misreading instruments, forgetting to zero an instrument, making calculation errors, mistakes in reporting units, etc. While you may occasionally make mistakes, they are *not* legitimate sources of *uncertainty*. They should be discovered and eliminated. With proper care you won't make them.

Systematic errors: Influences that produce consistent deviations in a set of measurements. For instance, they may result from using uncalibrated instruments. These are usually the hardest kind of errors to discover. We will say more about systematic errors when we discuss the evaluation of experimental discrepancies.

Example: Finding the mass of an object

Suppose you are measuring the mass “*m*” of an object on a triple-beam balance as shown in Fig. 1. First you make certain that the balance is “zeroed”—i.e., that the indicator points to zero when all the balance weights are set to their “zero” positions and there is no mass in the pan. (Even this preliminary procedure is part of the measurement process; it involves your judgment; and it is a source of uncertainty in the measurement.) Next you place the object to be measured in the pan and adjust the two larger calibrated masses on the measuring unit as shown in Fig. 1 to be near the balance condition. The two larger calibrated masses rest in indents assuring you that they are placed properly. (How *well* do they “assure you” of that? Would they be in *exactly* the same position if you were to make the measurement again?) Finally you adjust the smallest of the three calibrated masses until the pointer on the unit indicates that it is in balance. (Notice carefully that this calls, again, for a judgment on your part. You could *certainly* move the small mass by a little bit and still *consider* the scale to be “balanced.”)

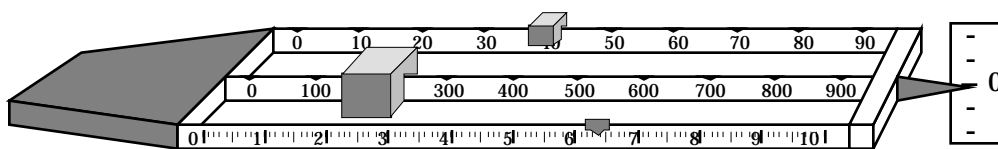


Figure 1: The measuring unit of a triple-beam balance

The best way to determine the uncertainty in a measurement is to make the measurement—starting from scratch—many times. Each *individual* measurement will be subject to the limitations imposed by the instrumental precision. By making a set of measurements, however, we will see if there is additional sample uncertainty over and above the instrumental uncertainty. As a result we will form a good idea about *both* the best value (m_{best}) and its uncertainty (m). For instance, suppose that you measured the mass of the same object six times with the balance beam and got the following values:

23.44 g	23.50 g	(Six single measurements of the mass, each subject to the same instrumental precision of about ± 0.01 g.)
23.31 g	23.35 g	
23.47 g	23.38 g	

Note the following:

- The values are all between 23.31 g and 23.50 g.
- Their average is 23.4083 g.
- They range fairly uniformly above and below this value by about 0.1 g.
- No values are particularly out of line with the rest as, say, 25.10 g would be.

A reasonable conclusion is that the measurements are “consistent” with each other and simply reflect the sample uncertainty that results from our judgment in the use of the scale. The measurements imply an uncertainty m of about 0.1 g. Given this, it makes no sense to report our best value with digits beyond the tenth’s place. Accordingly we might take 23.4 g for m_{best} and 0.1 g for m and report this concisely as

$$m = m_{\text{best}} \pm m = (23.4 \pm 0.1) \text{ g} = (2.34 \pm 0.01) \times 10^{-2} \text{ kg}$$

Another reasonable person might decide that the “true value” probably lies somewhat closer to the average than to the outlying values and might report the result as

$$m = (23.41 \pm 0.05) \text{ g} \quad (\text{Another } \textit{reasonable} \text{ way to report the result})$$

In any event observe the following rules:

Rules for reporting experimental values

- 1 Get a *reasonable* value for the uncertainty.
- 2 Round the *uncertainty* to *one significant digit*. (One possible exception to this rule: *If* that digit is a “1” or a “2” you *may* want to keep one *extra* digit. Rounding 0.14 to 0.1 is a 30% change; rounding 0.84 to 0.8 is only a 5% change!)
- 3 Round the *best value* to the *same decimal place* as the *single* digit in the uncertainty. (*If* you have kept two digits in the uncertainty, then round the best value to the same decimal place as the *second* digit in the uncertainty.)
- 4 Report the best value and its uncertainty with units and any common powers of ten factored out as shown in the examples above.

Nobody, can tell you the “right values” to report for the best value and its uncertainty; these involve your judgment. But, while there is no “right answer,” there are *definitely* wrong answers. In the current case, it would *definitely* be wrong to say any of the following

- | | |
|--------------------------------------|---|
| $m = 23.4083$ | (Wrong for lots of reasons, most notably because it doesn’t include units.) |
| $m = 23.4083 \text{ g}$ | (O.K., it has units, but it’s <i>still</i> wrong because it doesn’t give the uncertainty.) |
| $m = (23.4083 \pm 0.0001) \text{ g}$ | (O.K., now we have an uncertainty, but it’s <i>much</i> too small; any <i>reasonable</i> person would agree.) |
| $m = (23.4083 \pm 0.0736) \text{ g}$ | (A <i>reasonable</i> uncertainty reported with too many digits; <i>uncertainties</i> themselves are seldom, if ever, known that accurately.) |
| $m = (23.4083 \pm 0.07) \text{ g}$ | (A reasonably reported uncertainty, but the best value reports digits that are <i>not</i> significant.) |
| $m = (23 \pm 1) \text{ g}$ | (Correct agreement between the uncertainty and the best value, but the uncertainty is simply not that big; any <i>reasonable</i> person would agree.) |

Uncertainties in Calculations: “Propagation of Uncertainty”

Once we have our measured values and reasonable estimates of the uncertainties in them, we are ready to do the calculations that give us our experimental results. We need to obtain *reasonable* estimates for the *uncertainty* in those results. The process by which this is done is called “propagation of uncertainty.” The following “Brute Force” method determines the uncertainty in a result in a rather direct manner.

The “Brute Force” method for propagating uncertainties

Consider a quantity f that is to be calculated from measurements. Calculate f_{\max} and f_{\min} based on appropriate combinations of the maximum and minimum values for the measured quantities. From these determine f_{best} and f as follows:

$$f_{\text{best}} = \frac{1}{2}(f_{\max} + f_{\min}) \quad \text{and} \quad f = \frac{1}{2}(f_{\max} - f_{\min})$$

Although the Brute Force method *sounds* straightforward, it is *not* always easy to carry out in complex calculations. Therefore, we will also develop a set of three simple rules that can be used to propagate uncertainties through complex calculations one step at a time.

Uncertainties in Addition and Subtraction

Suppose that we want to add the two measured lengths

$$L_1 = (25 \pm 1) \text{ cm} \quad \text{and} \quad L_2 = (20 \pm 2) \text{ cm}$$

to get $L = L_1 + L_2$. Remember that “ 25 ± 1 ” means “somewhere between 24 and 26” and “ 20 ± 2 ” means “somewhere between 18 and 22.” The Brute Force method would have us find L_{\max} and L_{\min} like this:

$$L_{\max} = L_{1\max} + L_{2\max} = 26 \text{ cm} + 22 \text{ cm} = 48 \text{ cm}$$

and

$$L_{\min} = L_{1\min} + L_{2\min} = 24 \text{ cm} + 18 \text{ cm} = 42 \text{ cm}$$

In other words the range for L is between 42 cm and 48 cm. Accordingly, we would write

$$L = (45 \pm 3) \text{ cm}$$

Note, however, that there is a simpler method since it is no accident that $L = L_1 + L_2$. It is easy to show, in general, that when *adding or subtracting* numbers, the *absolute* uncertainties *add*. This is the first of our three rules for propagating uncertainties through calculations.

Uncertainties in Multiplication and Division

Next suppose that we want to divide the quantity

$$V = (100 \pm 3) \text{ cm}^3 \quad \text{by} \quad A = (20 \pm 1) \text{ cm}^2$$

to get $L = V/A$. It should be clear that

$$L_{\max} = V_{\max}/A_{\min} = (103 \text{ cm}^3)/(19 \text{ cm}^2) = 5.42 \text{ cm}$$

and

$$L_{\min} = V_{\min}/A_{\max} = (97 \text{ cm}^3)/(21 \text{ cm}^2) = 4.62 \text{ cm}$$

So we would get $L = (5.02 \pm 0.40) \text{ cm}$ which, after rounding appropriately, becomes

$$L = (5.0 \pm 0.4) \text{ cm}$$

Once again, however, there is a simpler method. Note that

$$\frac{L}{L_{\text{best}}} = \frac{V}{V_{\text{best}}} + \frac{A}{A_{\text{best}}}$$

$$\text{i.e., } \frac{0.4}{5.0} = \frac{3}{100} + \frac{1}{20} \quad \text{or} \quad 8\% = 3\% + 5\%$$

Is this an accident? In fact, no. It is relatively easy to show, in general, that when *multiplying or dividing* numbers, the *relative* uncertainties *add*. This is the second of our three rules.

Uncertainties in Powers

Finally, suppose that we want to take the square root of

$$A = (25 \pm 1) \text{ cm}^2$$

to get $L = A^{1/2}$. It should be clear from the Brute Force method that

$$L_{\max} = A_{\max}^{1/2} = (26 \text{ cm})^{1/2} = 5.099 \text{ cm}$$

and

$$L_{\min} = A_{\min}^{1/2} = (24 \text{ cm})^{1/2} = 4.899 \text{ cm}$$

So we would get $L = (4.999 \pm 0.100) \text{ cm}$ which, after rounding appropriately, becomes

$$L = (5.0 \pm 0.1) \text{ cm}$$

Once again, however, there is a simpler method. Note that

$$\frac{L}{L_{\text{best}}} = \frac{1}{2} \frac{A}{A_{\text{best}}}$$

$$\text{i.e., } \frac{0.1}{5.0} = \frac{1}{2} \frac{1}{25} \quad \text{or} \quad 2\% = \frac{1}{2} 4\%$$

Is *this* an accident? Again, no. It is relatively easy to show, in general, that when raising a number to a power, its *relative* uncertainty is multiplied by the power. This is the last of our three rules which we now summarize as the “step-by-step” method.

Step-by-Step method for propagating uncertainties*Rule #1—Addition and Subtraction*

When adding and/or subtracting a set of numbers, the *absolute* uncertainty in the result is equal to the *sum* of the *absolute* uncertainties in the set of numbers, i.e.,

If $f = w + x + \dots - y - z - \dots$

Then $f = w + x + \dots + y + z + \dots$

(Note: The uncertainties are *all added*, never subtracted.)

Rule #2—Multiplication and Division

When multiplying and/or dividing a set of numbers, the *relative* uncertainty in the result is equal to the *sum* of the *relative* uncertainties in the set of numbers, i.e.,

If $f = \frac{w \ x \ \dots}{y \ z \ \dots}$

Then $\frac{f}{f_{\text{best}}} = \frac{w}{w_{\text{best}}} + \frac{x}{x_{\text{best}}} + \dots + \frac{y}{y_{\text{best}}} + \frac{z}{z_{\text{best}}} + \dots$

(Note: The relative uncertainties are *all added*, never subtracted.)

Rule #3—Powers

When raising a number to a power, the *relative* uncertainty in the result is equal to the *relative* uncertainty in the original number *times* the power, i.e.,

If $f = x^n$

Then $\frac{f}{f_{\text{best}}} = n \frac{x}{x_{\text{best}}}$

An Example

Suppose that

$$r = (a - b/c)^2$$

where $a = (12.3 \pm 0.1)$ cm, $b = (72 \pm 2)$ cm³, and $c = (8.5 \pm 0.3)$ cm². Since the first thing we are going to do is a division, we need the *relative* uncertainties in b and c so we can add them. (Rule #2) We will *express* the relative uncertainties as percentages. For instance, the relative uncertainty in b is $\frac{2}{72} = .028 = 2.8\%$.

$$\begin{aligned} r &= (12.3 \pm 0.1) \text{ cm} - \frac{(72 \pm 2) \text{ cm}^3}{(8.5 \pm 0.3) \text{ cm}^2} \\ &= (12.3 \pm 0.1) \text{ cm} - \frac{72 \text{ cm}^3 \pm 2.8\%}{8.5 \text{ cm}^2 \pm 3.5\%} \\ &= [(12.3 \pm 0.1) \text{ cm} - (8.47 \text{ cm} \pm 6.3\%)]^2 \end{aligned}$$

(While we are doing the calculation we will keep “too many” digits in our numbers to avoid accumulating round off error. We will round off properly at the *end* of the entire calculation.)

Next we will do a subtraction. We need the *absolute* uncertainty in the last term so that we can add it to that of the first term. (Rule #1) We find that 6.3% of 8.47 cm is 0.53 cm so

$$\begin{aligned} r &= [(12.3 \pm 0.1) \text{ cm} - (8.47 \pm 0.53) \text{ cm}]^2 \\ &= [(3.83 \pm 0.63) \text{ cm}]^2 \end{aligned}$$

Next, we are going to square this value. We need to go *back* to the relative uncertainty again so that we can double it. (Rule #3)

$$\begin{aligned} r &= [3.83 \text{ cm} \pm 16.4\%]^2 \\ &= 14.7 \text{ cm}^2 \pm 33\% \end{aligned}$$

Finally, we express the result in standard form, rounding appropriately.

$$r = (15 \pm 5) \text{ cm}^2$$

Just to check, let's try the Brute Force method

$$\begin{aligned} r_{\min} &= [(12.2 - 74/8.2)\text{cm}]^2 = 10.1 \text{ cm}^2 \\ r_{\max} &= [(12.4 - 70/8.8)\text{cm}]^2 = 19.8 \text{ cm}^2 \end{aligned}$$

So we get $r = (14.95 \pm 4.85) \text{ cm}^2$ which, after rounding appropriately, becomes

$$r = (15 \pm 5) \text{ cm}^2$$

In agreement with the result from the Step-by-Step method

Evaluating the “Significance” of “Discrepancies”

Usually at the end of an experiment we will compare our results with those of others or the predictions of some theoretical calculation. This will involve comparing two or more values to see how well they agree with each other. Any difference between two such values is called a “discrepancy.” Keep in mind, however, that discrepancies are *inevitable* since the results of experimental work will always be subject to experimental uncertainty. The important question is, “*Is the discrepancy within the limits set by the experimental uncertainty?*” If it is, we say the discrepancy “is *not* significant;” that is, it is *accounted for* by the size of the experimental uncertainty. If the discrepancy is outside the limits set by the experimental uncertainty, we say the discrepancy “*is* significant;” that is, it is *not* accounted for by the size of the experimental uncertainty. (Please keep in mind that this is never an “all or nothing” judgment; “significant” discrepancies vary from “somewhat significant” to “extremely significant.”)

When the discrepancy is judged to be “significant” we have some more work to do. Without question, the *first* thing we must do is to check our calculations and make sure that we haven’t simply made a mistake.

If our calculations are correct we should consider the possibility that we have underestimated the experimental uncertainties. Generally this can only be done by taking more data and reconsidering our estimates of the uncertainty in our measurements. If we find that we have underestimated the uncertainties, a reanalysis of the data may render the discrepancy “insignificant.”

If we *still* find that the discrepancy is significant, we would want to take a look at the numbers with which we are comparing our results—especially if they are the result of theoretical calculations. The predictions of theory are only as reliable as the theory upon which they are based. For example, the predicted range R of a projectile shot horizontally with initial speed v off of a cliff with height h is

$$R = v \sqrt{\frac{2h}{g}} \quad (\text{where } g = \text{the acceleration due to gravity})$$

This prediction assumes, however, that 1) there is *no* air resistance, 2) the projectile is shot *exactly* horizontally, 3) h and R are measured between appropriate points—not a trivial task, 4) the acceleration due to gravity is *constant*, etc. Some of these assumptions are likely to be *very* good, but others may be highly questionable. Ignoring these factors introduces systematic errors into the experiment which, in turn, will produce non-random discrepancies in the results. When looking for this type of error pay attention to the algebraic sign of the observed discrepancy. In the case of the projectile, for instance, if we observe that our *experimental* range is consistently *larger* than the range predicted from *theory*, we can rule out air resistance as a possible explanation.

If we still judge the discrepancy significant after careful consideration of *every* thing that might *possibly* account for it *and* if we can convince others to do the same experiments, reproduce our discrepancies, and agree that they are significant (a *big* if!), then it *may* be time to celebrate. For the discovery of a *genuinely* significant discrepancy is an important event in an *experimental* science like physics. No matter how deeply we believe in a theory, in spite of how many other phenomena it explains, and, certainly, regardless of how many Nobel Prizes its developers may have won, if it *really* does not agree with the results of experiment, it is wrong.

Exercises:

- Each of the following lists represent a series of measurements of some quantity. From each list determine a “best value” and an associated *reasonable* uncertainty. (In each case you should first determine the apparent instrumental precision and sample uncertainty.)
 - 15.42 mm, 15.45 mm, 15.37 mm, 15.36 mm, 15.39 mm, 15.41 mm
 - 34.1 g, 34.1 g, 34.1 g, 34.1 g, 34.1 g
 - 2.44 s, 2.48 s, 2.37 s, 2.51 s, 2.42 s, 2.46 s, 2.44 s, 2.39 s, 2.49 s
- Perform the following calculations and express your result properly with uncertainty and units.
 - $(33.8 \pm 0.2) \text{ m} - (18.2 \pm 0.1) \text{ m}$ Answer: $(15.6 \pm 0.3) \text{ m}$
 - $\frac{2 [(12.4 \pm 0.3) \text{ cm}]}{(2.88 \pm 0.07) \times 10^{-3} \text{ s}}$ Answer: $(86 \pm 4) \text{ m/s}$
 - $[(7.44 \pm 0.04) \text{ cm}]^2$ Answer: $(174 \pm 2) \text{ cm}^2$
or $(1.74 \pm 0.02) \times 10^{-2} \text{ m}^2$
 - $\frac{(26.72 \pm 0.05) \text{ cm} - (13.8 \pm 0.2) \text{ cm}}{[(4.11 \pm 0.03) \text{ s}]^2}$ Answer: $(0.76 \pm 0.03) \text{ cm/s}^2$
or $(7.6 \pm 0.3) \times 10^{-3} \text{ m/s}^2$
or $(7.6 \pm 0.3) \text{ mm/s}^2$
 - $\frac{(13.31 \pm 0.04) \text{ cm} + (4.5 \pm 0.2) \text{ cm}}{(12.23 \pm 0.05) \text{ s}}$
 - $\sqrt[4]{[(4.65 \pm 0.03) \text{ mm}]^2 [(6.7 \pm 0.1) \text{ cm}]}$
 - $(9.80 \pm 0.01) \text{ m/s}^2 - \frac{2 [(88.5 \pm 0.3) \text{ cm}]}{[(1.78 \pm 0.04) \text{ s}]^2}$
- Which of the following pairs of numbers have significant discrepancies?
 - $(23.4 \pm 0.3) \text{ N}$ and 23.6 N
 - $(10 \pm 6) \text{ J}$ and 5 J
 - $(2.34467 \pm 0.00003) \text{ g}$ and $(2.34441 \pm 0.00001) \text{ g}$
 - $(1.56 \pm 0.03) \times 10^4 \text{ kg s}$ and $(4.41 \pm 0.03) \times 10^3 \text{ g hr}$