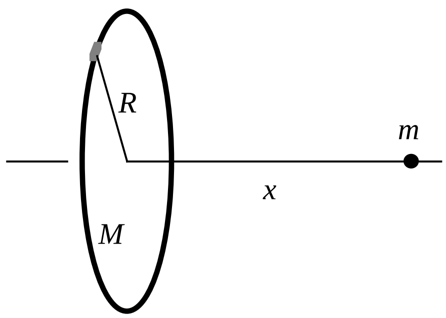
When and how do I “set up” an integral?

1. **Determine that an integral needs to be done.** If you pay attention to what you are doing you will always recognize the need to integrate due to the fact that some quantity in a formula you are trying to use has a variable value. For instance: $t = \frac{d}{v}$ applies to an object moving at constant speed, but what if the speed is not constant over the distance *d*? In this case there is no obvious single value of *v* to use. Similarly $F = \frac{Gm_1 m_2}{d^2}$ applies to two point objects separated by a distance *d*, but what if one or both objects have spatial extent? In this case there is no obvious value of *d* to use.

Case 1:

The gravitational force due to a uniform ring of radius *R* and mass *M* on a particle of mass *m* located on its axis at a distance *x* from its center.

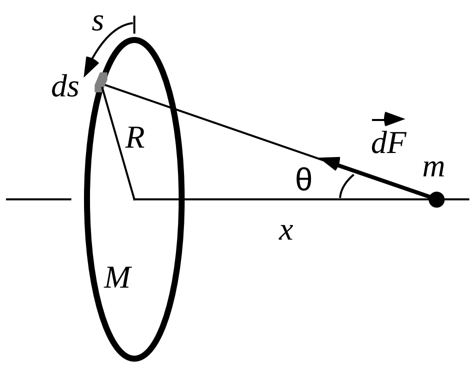


Case 2:

The time required for an object to travel a distance *d* from rest if $v = a + bx + cx^2$ where *a* and *b* are constants and *x* is the distance from the original location.

2. **Find the infinitesimal contribution to the result from an infinitesimal portion.** This will require choosing a coordinate that will become the variable of integration.

Case 1:

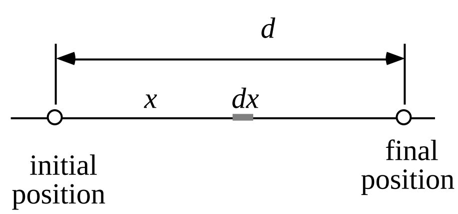


$d\vec{F} = \frac{G\ m\ d\ M}{d^2}$, at an angle θ from the axis

where $d = \left(R^2 + x^2\right)^{1/2}$ and $\frac{dM}{M} = \frac{ds}{2\pi R}$ so

$d\vec{F} = \frac{Gm\left(\frac{Mds}{2\pi R}\right)}{R^2 + x^2}$, at an angle θ from the axis

Case 2:



$dt = \frac{dx}{v} = \frac{dx}{a + bx + cx^2}$

3. **Add 'em up!**

Case 1:

Since we are adding vectors, we need to add the components separately. Due to the symmetry of the problem we see, however, that the off axis components will result in a *net* off axis force component of zero. Thus we need only to add the *on* axis components

$$dF_{||} = \frac{G\ m\left(\frac{Mds}{2\pi R}\right)}{R^2 + x^2}\cos\theta = \frac{G\ m\left(\frac{Mds}{2\pi R}\right)}{R^2 + x^2}\frac{x}{\left(R^2 + x^2\right)^{1/2}}$$

$$= \frac{GmM}{2\pi R}\frac{x\ d\ s}{\left(R^2 + x^2\right)^{3/2}}$$

so that

$$F_{||} = \int dF_{||} = \int_0^{2\pi R}\frac{GmM}{2\pi R}\frac{x\ ds}{\left(R^2 + x^2\right)^{3/2}}$$

$$= \frac{GmM}{2\pi R}\frac{x}{\left(R^2 + x^2\right)^{3/2}}\int_0^{2\pi R} ds$$

$$= \frac{GmM\ x}{\left(R^2 + x^2\right)^{3/2}}$$

Notice that, in this case, the integrand was *completely* independent of the variable of integration so the integral itself was completely trivial!

Case 2:

We are all ready to go!

$$t = \int dt = \int_0^d \frac{dx}{a + bx + cx^2}$$

$$= \frac{2}{\sqrt{4ac - b^2}}\left[\arctan\left(\frac{2cd + b}{\sqrt{4ac - b^2}}\right) - \arctan\left(\frac{b}{\sqrt{4ac - b^2}}\right)\right]$$

Notice that, in this case, while the set up was easy, the integral *itself* was very complicated!