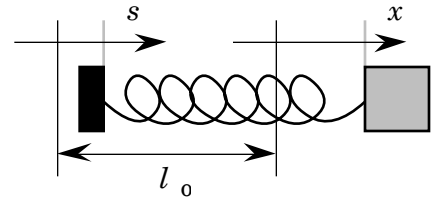


The Sinusoidally Forced, Linearly Damped, Simple Harmonic Oscillator

A mass m is attached to one end of a spring with unstretched length l_0 and force constant k . The other end is driven back and forth sinusoidally as $s(t) = s_0 \sin(\omega t)$. In addition the mass is subject to a viscous linear drag force with drag coefficient b . The equation of motion for the system is, therefore,



$$m \frac{d^2x}{dt^2} = -k(x - s(t)) - b \frac{dx}{dt} \quad (1)$$

or

$$m \ddot{x} + b \dot{x} + kx = ks_0 \sin(\omega t). \quad (2)$$

We can cast this in a more general form which serves to characterize *any* sinusoidally driven, linearly damped, simple harmonic oscillator.

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \omega_0^2 s_0 \sin(\omega t) \quad (3)$$

where γ is called the “decay constant” and ω_0 is called the “natural frequency” of the system. (In our *specific* case, we would find that $\gamma = b/(2m)$ and $\omega_0 = \sqrt{k/m}$.)

Equation (3) is an inhomogeneous, second order, linear differential equation and has different solutions according to whether $\gamma < \omega_0$ (the so-called “underdamped” case), $\gamma > \omega_0$ (the “overdamped” case), or $\gamma = \omega_0$ (the “critically damped” case). We will restrict our attention to the underdamped case for which the solution is

$$x(t) = A e^{-\gamma t} \cos(\omega' t + \phi_0) + x_{\max} \sin(\omega t - \phi) \quad (4a)$$

where

$$\omega' = \sqrt{\omega_0^2 - \gamma^2} \quad (4b)$$

$$x_{\max} = \frac{s_0}{\sqrt{[1 - (\omega/\omega_0)^2]^2 + \frac{\gamma^2}{\omega_0^2}}} \quad (4c)$$

$$\phi = \arctan \frac{\gamma/\omega_0}{1 - (\omega/\omega_0)^2} \quad (\text{with } 0 < \phi < \pi) \quad (4d)$$

$$Q = \omega_0 / (2\gamma) \quad (\text{the so-called “quality factor” of the oscillator}) \quad (4e)$$

and A and ϕ_0 are constants of integration determined by the initial conditions

$x_0 = x(0)$ and $v_0 = \frac{dx}{dt}(0)$ from

$$\phi_0 = \arctan \frac{1}{\gamma} \frac{x_{\max} \cos \phi - v_0}{x_{\max} \sin \phi + x_0} \quad (4f)$$

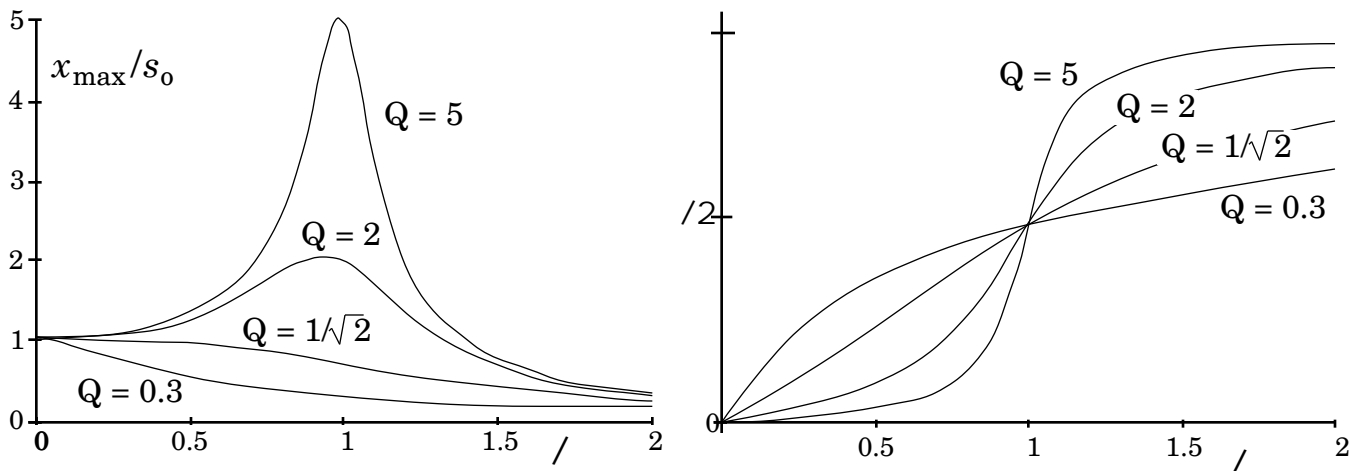
$$A = \frac{x_0 + x_{\max} \sin \phi}{\cos \phi_0} \quad (4g)$$

There, now that all *that's* done, let's see if we can regain a glimpse of the forest through all of these apparent trees. In fact, things aren't *nearly* as horrible as they might look.

- Case 1: Undriven ($s_0 = 0$). In this case, the second term of Eq. (4a) vanishes. Furthermore, we usually don't really care about any detailed matching of initial conditions (and, even if we did, Eqs. (4f) and (4g) are rendered substantially simpler by the fact that $x_{\max} = 0$.)

The important points, then, are quite simple: (1) The system oscillates at a frequency ω which is, according to Eq. (4b), somewhat less than the "natural frequency." (2) The amplitude of the oscillations dies off exponentially as $e^{-\gamma t}$.

- Case 2: Driven ($s_0 > 0$). In this case, we usually care only about the so-called "steady-state" oscillations which are given by the second term of Eq. (4a). Furthermore, this term is correct as written whether the oscillator is underdamped, critically damped, or overdamped. The solution is now fully characterized by two quantities— x_{\max}/s_0 and ϕ as given by Eqs. (4c – e). These are referred to as the amplitude and phase "response" of the system and it is evident that, for a given ratio of the driving frequency to the natural frequency, *they depend only on the quality factor Q of the system* as shown in the frequency response curves reproduced below.



The most important characteristics of these plots are listed below.

- Amplitude response: A peak appears for all values of $Q > 1/\sqrt{2}$. (Note that for *critical* damping $Q = 1/2$.) The peak value is $(x_{\max}/s_0)_{\text{peak}} = Q\sqrt{1 - 1/(4Q^2)}$ and it occurs at a normalized frequency $\omega/\omega_0 = \sqrt{1 - 1/(2Q^2)}$. **The peak becomes taller and sharper as Q increases.** Regardless of the value of Q , at low frequencies, $x_{\max} \approx s_0$ and at high frequencies $x_{\max} \approx (s_0/\omega)^2$.
- Phase response: At low frequencies the system oscillates approximately in phase with the driver. At the natural frequency it lags by 1/4 cycle. At high frequencies it oscillates out of phase. Roughly speaking, the transition between the low and high frequency behaviors takes place in a region of drive frequencies that is centered around the natural frequency and that varies inversely with Q . **The phase transition becomes more abrupt as Q increases.**