Physics 409 **Problem #3—Nonlinear Dynamics and Chaos** Fall 2002

Consider a simple pendulum subject to linear, velocity-dependent drag, and a sinusoidal driving force. As shown in class, the equation of motion is a nonlinear, second order differential equation and can be written as,

$$\ddot{x} = -\sin(x) - \gamma\dot{x} + f\sin(\Omega t)$$

Your assignment is to:

1. Write a program that will solve for the motion of the particle—given x_o, v_o [$\equiv (\dot{x})_o$], and the values of the three parameters in the force law (γ, f, and Ω)—using the second order Runge-Kutta method presented in class.

2. Have your program make a plot of the particle's phase-space trajectory—that is, a plot of v [$\equiv \dot{x}$] versus x. Have it mark points on the trajectory that correspond to the same *phase* of the driving force (*i.e.*, "Poincaré points") with a different color than you use for the rest of the points. Include an option to plot *only* the Poincaré points.

3. Use your program to analyze *at least* the following situations:

   A. $\gamma = 0$, $f = 0$

   This is the undamped, undriven case. Look at what happens for various initial velocities starting at $x = 0$. What happens when the initial velocity gets "too large"?

   B. $\gamma \geq 0$, $f = 0$

   This is the damped undriven case. Again look at what happens for various initial velocities. What happens as γ increases? What value of γ seems to correspond best to "critical damping"? Does it depend on the initial velocity?

   C. $\gamma = 0.4$, $f = 0.1$ or less, $\Omega = 0.5$ to 1.5

   This is the damped driven case with amplitudes low enough (because of large drag *and* small drive amplitude) that it can almost be considered a simple harmonic oscillator. Observe the transient behavior of the phase-space trajectory and the approach to the stable limit cycle—*i.e.*, the "steady state." What value of Ω maximizes the range of *velocities* that the particle attains in this limit cycle? What value maximizes the range of *positions* that the particle attains? What happens to the location of the specially-colored phase markers as you alter the frequency? How does all this square with the analytical results for the damped, driven, simple harmonic oscillator?

   D. $\gamma = 0.1$, $f = 1$ or more, $\Omega = 0.5$ to 1.5

   Now the amplitudes can get large enough that the nonlinear effects become quite important. What is the effect of the nonlinearity on the resonance behavior that you observed in part B? The pendulum is an example of what is sometimes called a "soft" oscillator because the restoring force increases *less* than linearly with x. As a result, you should see the resonance peak shift toward lower frequencies at higher drive amplitudes. You *may* also see the phenomenon called "hysteresis" which happens when more than one stable limit cycle exists in and the particular one that *is* reached depends on the details of how the system parameters have been varied in the past!

   E. Arbitrary parameters

   With the frequency set at some value and the drag relatively low, vary the drive amplitude slowly and look for bifurcations, period doublings, multiple attractors, and the onset of chaos. In a chaotic regime, watch the development of the Poincaré section for the strange attractor. There is no limit to the interesting behaviors that occur in the enormous parameter space of this problem. Use your program to do some research.

4. Different position dependences (optional, but not hard and lotsa fun!)

   Replace the nonlinear "$-\sin x$" restoring force with a linear Hooke's law "$-x$" restoring force (derived from the potential $\frac{1}{2}x^2$) or with an even *more* interesting *nonlinear* force derived from the "double well" potential $\frac{q}{4}x^4 - \frac{1}{2}x^2$. (Sketch a plot of $U(x)$ to see why it is called a "double well.") The Hooke's law case can be completely solved analytically and you can compare your results to the analytical solution. The double well case can only be solved numerically, but you can still understand some of the behavior in terms of the potential energy function.